By the Same Author

PROJECTIVE GEOMETRY
Blaisdell, Waltham, Mass.

THE REAL PROJECTIVE PLANE
Cambridge University Press

NON-EUCLIDEAN GEOMETRY
University of Toronto Press

REGULAR POLYTOPES
Macmillan, New York

TWELVE GEOMETRIC ESSAYS
Southern Illinois University Press

Introduction to

GEOMETRY

second edition

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Preface

I am grateful to the readers of the first edition who have made suggestions for improvement. Apart from some minor corrections, the principal changes are as follows.

The equation connecting the curvatures of four mutually tangent circles, now known as the *Descartes Circle Theorem* (p. 12), is proved along the lines suggested by Mr. Beecroft on pp. 91–96 of "The Lady's and Gentleman's Diary for the year of our Lord 1842, being the second after Bissextile, designed principally for the amusement and instruction of Students in Mathematics: comprising many useful and entertaining particulars, interesting to all persons engaged in that delightful pursuit."

For *similarity* in the plane, a new treatment (pp. 73–76) was suggested by A. L. Steger when he was a sophomore at the University of Toronto. For similarity in space, a different treatment (p. 103) was suggested by Professor Maria Wonenburger. A new exercise on p. 90 introduces the useful concept of *inversive distance*. Another has been inserted on p. 127 to exhibit R. Krasnodębski's drawings of symmetrical loxodromes.

Pages 203–208 have been revised so as to clarify the treatment of *affinities* (which preserve collinearity) and *equiaffinities* (which preserve area). The new material includes some challenging exercises. For the discovery of *finite* geometries (p. 237), credit has been given to von Staudt, who anticipated Fano by 36 years.

Page 395 records the completion, in 1968, by G. Ringel and J. W. T. Youngs, of a project begun by Heawood in 1890. The result is that we now know, for every kind of surface except the sphere (or plane), the minimal number of colors that will suffice for coloring every map on the surface.

Answers are now given for practically all the exercises; a separate booklet is no longer needed. One of the prettiest answers (p. 453) was kindly supplied by Professor P. Szász of Budapest.

H.S.M. Coxeter

Toronto, Canada
January, 1969

Preface to the first edition

For the last thirty or forty years, most Americans have somehow lost interest in geometry. The present book constitutes an attempt to revitalize this sadly neglected subject.

The four parts correspond roughly to the four years of college work. However, most of Part II can be read before Part I, and most of Part IV before Part III. The first eleven chapters (that is, Parts I and II) will provide a course for students who have some knowledge of Euclid and elementary analytic geometry but have not yet made up their minds to specialize in mathematics, or for enterprising high school teachers who wish to see what is happening just beyond their usual curriculum. Part III deals with the foundations of geometry, including projective geometry and hyperbolic non-Euclidean geometry. Part IV introduces differential geometry, combinatorial topology, and four-dimensional Euclidean geometry.

In spite of the large number of cross references, each of the twenty-two chapters is reasonably self-contained; many of them can be omitted on first reading without spoiling one's enjoyment of the rest. For instance, Chapters 1, 3, 6, 8, 13, and 17 would make a good short course. There are relevant exercises at the end of almost every section; the hardest of them are provided with hints for their solution. (Answers to some of the exercises are given at the end of the book. Answers to many of the remaining exercises are provided in a separate booklet, available from the publisher upon request.) The unifying thread that runs through the whole work is the idea of a group of transformations or, in a single word, *symmetry*.

The customary emphasis on analytic geometry is likely to give students the impression that geometry is merely a part of algebra or of analysis. It is refreshing to observe that there are some important instances (such as the Argand diagram described in Chapter 9) in which geometrical ideas are needed as essential tools in the development of these other branches of mathematics. The scope of geometry was spectacularly broadened by Klein in his *Erlanger Programm* (Erlangen program) of 1872, which stressed the fact that, besides plane and solid Euclidean geometry, there are many other geometries equally worthy of attention. For instance, many of Euclid's own propositions belong to the wider field of *affine* geometry, which is valid not

only in ordinary space but also in Minkowski's space-time, so successfully exploited by Einstein in his special theory of relativity.

Geometry is useful not only in algebra, analysis, and cosmology, but also in kinematics and crystallography (where it is associated with the theory of groups), in statistics (where finite geometries help in the design of experiments), and even in botany. The subject of topology (Chapter 21) has been developed so widely that it now stands on its own feet instead of being regarded as part of geometry; but it fits into the Erlangen program, and its early stages have the added appeal of a famous unsolved problem: that of deciding whether every possible map can be colored with four colors.

The material grew out of courses of lectures delivered at summer institutes for school teachers and others at Stillwater, Oklahoma; Lunenburg, Nova Scotia; Ann Arbor, Michigan; Stanford, California; and Fredericton, New Brunswick, along with several public lectures given to the Friends of Scripta Mathematica in New York City by invitation of the late Professor Jekuthiel Ginsburg. The most popular of these separate lectures was the one on the golden section and phyllotaxis, which is embodied in Chapter 11.

Apart from the general emphasis on the idea of transformation and on the desirability of spending some time in such unusual environments as affine space and absolute space, the chief novelties are as follows: a simple treatment of the orthocenter (§ 1.6); the use of dominoes to illustrate six of the seventeen space groups of two-dimensional crystallography (§ 4.4); a construction for the invariant point of a dilative reflection (§ 5.6); a description of the general circle-preserving transformation (§ 6.7) and of the spiral similarity (§ 7.6); an "explanation" of phyllotaxis (§ 11.5); an "ordered" treatment of Sylvester's problem (§ 12.3); an economical system of axioms for affine geometry (§ 13.1); an "absolute" treatment of rotation groups (§ 15.4); an elementary treatment of the horosphere (§ 16.8) and of the extreme ternary quadratic form (§ 18.4); the correction of a prevalent error concerning the shape of the monkey saddle (§ 19.8); an application of geodesic polar coordinates to the foundations of hyperbolic trigonometry (§ 20.6); the classification of regular maps on the sphere, projective plane, torus, and Klein bottle (§ 21.3); and the suggestion of a statistical honeycomb (§ 22.5).

I offer sincere thanks to M. W. Al-Dhahir, J. J. Burckhardt, Werner Fenchel, L. M. Kelly, Peter Scherk, and F. A. Sherk for critically reading various chapters; also to H G. Forder, Martin Gardner, and C. J. Scriba for their help in proofreading, to S. H. Gould, J. E. Littlewood, and J. L. Synge for permission to quote certain passages from their published works, and to M. C. Escher, I. Kitrosser, and the Royal Society of Canada for permission to reproduce the plates.

H.S.M. Coxeter

Toronto, Canada
March, 1961

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Plates

Mathematics possesses not only truth, but supreme beauty
—a beauty cold and austere, like that of sculpture,
without appeal to any part of our weaker nature . . .
sublimely pure, and capable of a stern perfection
such as only the greatest art can show.

BERTRAND RUSSELL (1872 - 1970)

Part I

1
Triangles

In this chapter we review some of the well-known propositions of elementary geometry, stressing the role of symmetry. We refer to Euclid's propositions by his own numbers, which have been used throughout the world for more than two thousand years. Since the time of F. Commandino (1509–1575), who translated the works of Archimedes, Apollonius, and Pappus, many other theorems in the same spirit have been discovered. Such results were studied in great detail during the nineteenth century. As the present tendency is to abandon them in favor of other branches of mathematics, we shall be content to mention a few that seem particularly interesting.

1.1 EUCLID

Euclid's work will live long after all the text-books of the present day are superseded and forgotten. It is one of the noblest monuments of antiquity.

Sir Thomas L. Heath (1861-1940)*

About 300 B.C., Euclid of Alexandria wrote a treatise in thirteen books called the *Elements*. Of the author (sometimes regrettably confused with the earlier philosopher, Euclid of Megara) we know very little. Proclus (410–485 A.D.) said that he "put together the Elements, collecting many of Eudoxus's theorems, perfecting many of Theaetetus's, and also bringing to irrefragable demonstration the things which were only somewhat loosely proved by his predecessors. This man lived in the time of the first Ptolemy, [who] once asked him if there was in geometry any shorter way than that of the Elements, and he answered that there was no royal road to geometry." Heath quotes a story by Stobaeus, to the effect that someone who had begun to read geometry with Euclid asked him "What shall I get by learning these things?" Euclid called his slave and said "Give him a dime, since he must make gain out of what he learns."

* Heath **1**, p. vi. (Such references are collected at the end of the book, pp. 415–417.)

Of the thirteen books, the first six may be very briefly described as dealing respectively with triangles, rectangles, circles, polygons, proportion, and similarity. The next four, on the theory of numbers, include two notable achievements: IX.2 and X.9, where it is proved that there are infinitely many prime numbers, and that $\sqrt{2}$ is irrational [Hardy **2**, pp. 32–36]. Book XI is an introduction to solid geometry, XII deals with pyramids, cones, and cylinders, and XIII is on the five regular solids.

According to Proclus, Euclid "set before himself, as the end of the whole Elements, the construction of the so-called Platonic figures." This notion of Euclid's purpose is supported by the Platonic theory of a mystical correspondence between the four solids

$$\left.\begin{array}{l}\text{cube,}\\ \text{tetrahedron,}\\ \text{octahedron,}\\ \text{icosahedron}\end{array}\right\}\text{and the four "elements"}\left\{\begin{array}{l}\text{earth,}\\ \text{fire,}\\ \text{air,}\\ \text{water}\end{array}\right.$$

[cf. Coxeter **1**, p. 18]. Evidence to the contrary is supplied by the arithmetical books VII–X, which were obviously included for their intrinsic interest rather than for any application to solid geometry.

1.2 PRIMITIVE CONCEPTS AND AXIOMS

"When I use a word," Humpty-Dumpty said, "it means just what I choose it to mean—neither more nor less."

Lewis Carroll (1832–1898)

[Dodgson **2**, Chap. 6]

In the logical development of any branch of mathematics, each definition of a concept or relation involves other concepts and relations. Therefore the only way to avoid a vicious circle is to allow certain *primitive* concepts and relations (usually as few as possible) to remain undefined [Synge **1**, pp. 32–34]. Similarly, the proof of each proposition uses other propositions, and therefore certain primitive propositions, called *postulates* or *axioms*, must remain unproved. Euclid did not specify his primitive concepts and relations, but was content to give definitions in terms of ideas that would be familiar to everybody. His five Postulates are as follows:

1.21 *A straight line may be drawn from any point to any other point.*

1.22 *A finite straight line may be extended continuously in a straight line.*

1.23 *A circle may be described with any center and any radius.*

1.24 *All right angles are equal to one another.*

1.25 *If a straight line meets two other straight lines so as to make the two interior angles on one side of it together less than two right angles, the other*

*straight lines, if extended indefinitely, will meet on that side on which the angles are less than two right angles.**

It is quite natural that, after a lapse of about 2250 years, some details are now seen to be capable of improvement. (For instance, Euclid I.1 constructs an equilateral triangle by drawing two circles; but how do we know that these two circles will intersect?) The marvel is that so much of Euclid's work remains perfectly valid. In the modern treatment of his geometry [see, for instance, Coxeter **3,** pp. 161–187], it is usual to recognize the primitive concept *point* and the two primitive relations of *intermediacy* (the idea that one point may be between two others) and *congruence* (the idea that the distance between two points may be equal to the distance between two other points, or that two line segments may have the same length). There are also various versions of the axiom of *continuity,* one of which says that every convergent sequence of points has a limit.

Euclid's "principle of superposition," used in proving I.4, raises the question whether a figure can be moved without changing its internal structure. This principle is nowadays replaced by a further explicit assumption such as the axiom of "the rigidity of a triangle with a tail" (Figure 1.2*a*):

1.26 *If ABC is a triangle with D on the side BC extended, while D' is analogously related to another triangle $A'B'C'$, and if $BC = B'C'$, $CA = C'A'$, $AB = A'B'$, $BD = B'D'$, then $AD = A'D'$.*

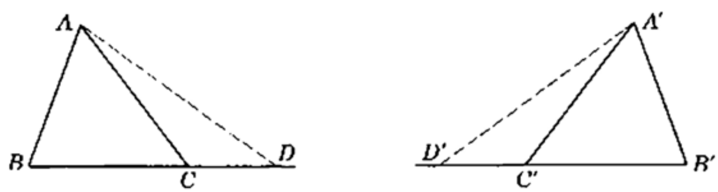


Figure 1.2a

This axiom can be used to extend the notion of congruence from line segments to more complicated figures such as angles, so that we can say precisely what we mean by the relation

$$\angle ABC = \angle A'B'C'.$$

Then we no longer need the questionable principle of superposition in order to prove Euclid I.4:

If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal sides equal, they will also have their third sides equal, and their remaining angles equal respectively; in fact, they will be congruent triangles.

* In Chapter 15 we shall see how far we can go without using this unpleasantly complicated Fifth Postulate.

1.3 PONS ASINORUM

> *Minos: It is proposed to prove I.5 by taking up the isosceles Triangle, turning it over, and then laying it down again upon itself.*
>
> *Euclid: Surely that has too much of the Irish Bull about it, and reminds one a little too vividly of the man who walked down his own throat, to deserve a place in a strictly philosophical treatise?*
>
> *Minos: I suppose its defenders would say that it is conceived to leave a trace of itself behind, and that the reversed Triangle is laid down upon the trace so left.*
>
> C. L. Dodgson (1832-1898)
>
> [Dodgson **3**, p. 48]

I.5. *The angles at the base of an isosceles triangle are equal.*

The name *pons asinorum* for this famous theorem probably arose from the bridgelike appearance of Euclid's figure (with the construction lines required in his rather complicated proof) and from the notion that anyone unable to cross this bridge must be an ass. Fortunately, a far simpler proof was supplied by Pappus of Alexandria about 340 A.D. (Figure 1.3*a*):

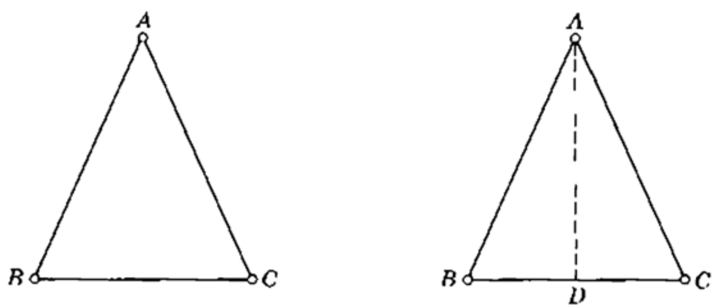


Figure 1.3a

Let ABC be an isosceles triangle with AB equal to AC. Let us conceive this triangle as two triangles and argue in this way. Since $AB = AC$ and $AC = AB$, the two sides AB, AC are equal to the two sides AC, AB. Also the angle BAC is equal to the angle CAB, for it is the same. Therefore all the corresponding parts (of the triangles ABC, ACB) are equal. In particular,

$$\angle ABC = \angle ACB.$$

The pedagogical difficulty of comparing the isosceles triangle ABC with itself is sometimes avoided by joining the apex A to D, the midpoint of the base BC. The median AD may be regarded as a *mirror* reflecting B into C. Accordingly, we say that an isosceles triangle is symmetrical by *reflection*, or that it has *bilateral symmetry*. (Of course, the idealized mirror used in geometry has no thickness and is silvered on both sides, so that it not only reflects B into C but also reflects C into B.)

Any figure, however irregular its shape may be, yields a symmetrical figure when we place it next to a mirror and waive the distinction between object and image. Such bilateral symmetry is characteristic of the external shape of most animals.

Given any point P on either side of a geometrical mirror, we can construct its reflected image P' by drawing the perpendicular from P to the mirror and extending this perpendicular line to an equal distance on the other side, so that the mirror perpendicularly bisects the line segment PP'. Working in the plane (Figure 1.3*b*) with a line AB for mirror, we draw two circles with centers A, B and radii AP, BP. The two points of intersection of these circles are P and its image P'.

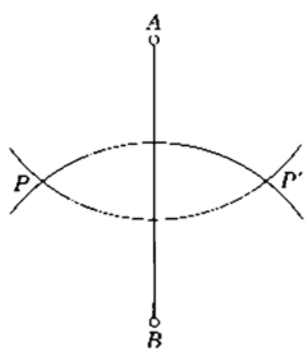


Figure 1.3b

We shall find that many geometrical proofs are shortened and made more vivid by the use of reflections. But we must remember that this procedure is merely a short cut: every such argument could have been avoided by means of a circumlocution involving congruent triangles. For instance, the above construction is valid because the triangles ABP, ABP' are congruent.

Pons asinorum has many useful consequences, such as the following five:

III.3. *If a diameter of a circle bisects a chord which does not pass through the center, it is perpendicular to it; or, if perpendicular to it, it bisects it.*

III.20. *In a circle the angle at the center is double the angle at the circumference, when the rays forming the angles meet the circumference in the same two points.*

III.21. *In a circle, a chord subtends equal angles at any two points on the same one of the two arcs determined by the chord* (e.g., in Figure 1.3*c*, $\angle PQQ' = \angle PP'Q'$).

III.22. *The opposite angles of any quadrangle inscribed in a circle are together equal to two right angles.*

III.32. *If a chord of a circle be drawn from the point of contact of a tangent, the angle made by the chord with the tangent is equal to the angle subtended by the chord at a point on that part of the circumference which lies on the far side of the chord* (e.g., in Figure 1.3*c*, $\angle OTP' = \angle TPP'$).

We shall also have occasion to use two familiar theorems on similar triangles:

VI.2. *If a straight line be drawn parallel to one side of a triangle, it will cut the other sides proportionately; and, if two sides of the triangle be cut proportionately, the line joining the points of section will be parallel to the remaining side.*

VI.4. *If corresponding angles of two triangles are equal, then corresponding sides are proportional.*

Combining this last result with III.21 and 32, we deduce two significant properties of secants of a circle (Figure 1.3c):

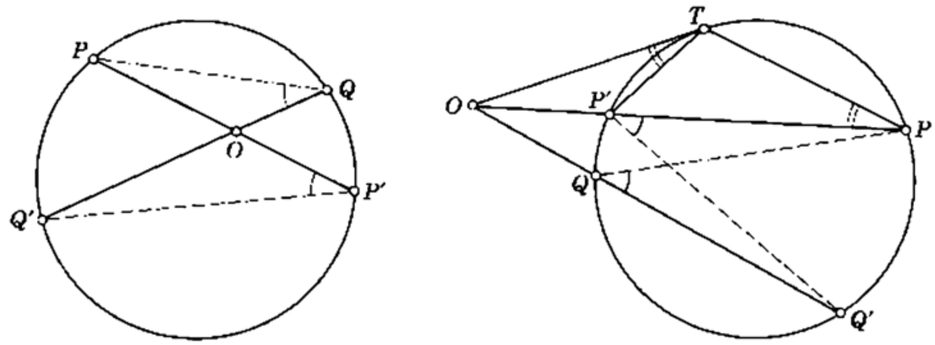


Figure 1.3c

III.35. *If in a circle two straight lines cut each other, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other* (i.e., $OP \times OP' = OQ \times OQ'$).

III.36. *If from a point outside a circle a secant and a tangent be drawn, the rectangle contained by the whole secant and the part outside the circle will be equal to the square on the tangent* (i.e., $OP \times OP' = OT^2$).

Book VI also contains an important property of area:

VI.19. *Similar triangles are to one another in the squared ratio of their corresponding sides* (i.e., if ABC and $A'B'C'$ are similar triangles, their areas are in the ratio $AB^2 : A'B'^2$).

This result yields the following easy proof for the theorem of Pythagoras [see Heath **1**, p. 353; **2**, pp. 210, 232, 269]:

I.47. *In a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the two catheti.*

In the triangle ABC, right-angled at C, draw CF perpendicular to the hypotenuse AB, as in Figure 1.3d. Then we have three similar right-angled triangles ABC, ACF, CBF, with hypotenuses AB, AC, CB. By VI.19, the areas satisfy

$$\frac{ABC}{AB^2} = \frac{ACF}{AC^2} = \frac{CBF}{CB^2}.$$

Evidently, $ABC = ACF + CBF$. Therefore $AB^2 = AC^2 + CB^2$.

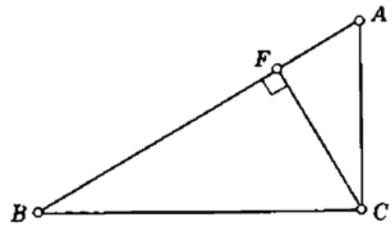


Figure 1.3d

EXERCISES

1. Using rectangular Cartesian coordinates, show that the reflection in the y-axis ($x = 0$) reverses the sign of x. What happens when we reflect in the line $x = y$?

2. Deduce I.47 from III.36 (applied to the circle with center A and radius AC).

3. Inside a square $ABDE$, take a point C so that CDE is an isosceles triangle with angles 15° at D and E. What kind of triangle is ABC?

4. Prove the Erdös-Mordell theorem: If O is any point inside a triangle ABC and P, Q, R are the feet of the perpendiculars from O upon the respective sides BC, CA, AB, then

$$OA + OB + OC \geqslant 2(OP + OQ + OR).$$

(*Hint:** Let P_1 and P_2 be the feet of the perpendiculars from R and Q upon BC. Define analogous points Q_1 and Q_2, R_1 and R_2 on the other sides. Using the similarity of the triangles PRP_1 and OBR, express P_1P in terms of RP, OR, and OB. After substituting such expressions into

$$OA + OB + OC \geqslant OA(P_1P + PP_2)/RQ + OB(Q_1Q + QQ_2)/PR \\ + OC(R_1R + RR_2)/QP,$$

collect the terms involving OP, OQ, OR, respectively.)

5. Under what circumstances can the sign $\geqslant$ in Ex. 4 be replaced by $=$?

6. In the notation of Ex. 4,

$$OA \times OB \times OC \geqslant (OQ + OR)(OR + OP)(OP + OQ).$$

(A. Oppenheim, *American Mathematical Monthly*, **68** (1961), p. 230. See also L. J. Mordell, *Mathematical Gazette*, **46** (1962), pp. 213–215.)

7. Prove the Steiner-Lehmus theorem: Any triangle having two equal internal angle bisectors (each measured from a vertex to the opposite side) is isosceles. (*Hint:*† If a triangle has two different angles, the smaller angle has the longer internal bisector.)

* Leon Bankoff, *American Mathematical Monthly*, **65** (1958), p. 521. For other proofs see G. R. Veldkamp and H. Brabant, *Nieuw Tijdschrift voor Wiskunde*, **45** (1958), pp. 193–196; **46** (1959), p. 87.

† Court **2**, p. 72. For Lehmus's proof of 1848, see Coxeter and Greitzer **1**, p. 15.

1.4 THE MEDIANS AND THE CENTROID

Oriental mathematics may be an interesting curiosity, but Greek mathematics is the real thing. . . . The Greeks, as Littlewood said to me once, are not clever schoolboys or "scholarship candidates," but "Fellows of another college." So Greek mathematics is "permanent," more permanent even than Greek literature. Archimedes will be remembered when Aeschylus is forgotten, because languages die and mathematical ideas do not.

G. H. Hardy (1877-1947)
[Hardy **2**, p. 21]

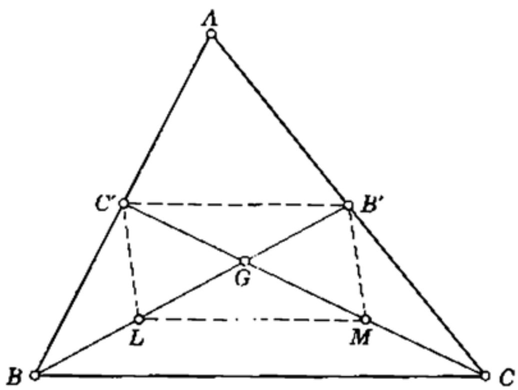


Figure 1.4a

The line joining a vertex of a triangle to the midpoint of the opposite side is called a *median*.

Let two of the three medians, say BB' and CC', meet in G (Figure 1.4*a*). Let L and M be the midpoints of GB and GC. By Euclid VI.2 and 4 (which were quoted on page 8), both $C'B'$ and LM are parallel to BC and half as long. Therefore $B'C'LM$ is a parallelogram. Since the diagonals of a parallelogram bisect each other, we have

$$B'G = GL = LB, \qquad C'G = GM = MC.$$

Thus the two medians BB', CC' trisect each other at G. In other words, this point G, which could have been defined as a point of trisection of one median, is also a point of trisection of another, and similarly of the third. We have thus proved [by the method of Court **1**, p. 58] the following theorem:

1.41 *The three medians of any triangle all pass through one point.*

This common point G of the three medians is called the *centroid* of the triangle. Archimedes (*c.* 287–212 B.C.) obtained it as the center of gravity of a triangular plate of uniform density.

EXERCISES

1. Any triangle having two equal medians is isosceles.*
2. The sum of the medians of a triangle lies between $\frac{3}{4}$p and p, where p is the sum of the sides. [Court **1**, pp. 60–61.]

1.5 THE INCIRCLE AND THE CIRCUMCIRCLE

Alone at nights,
I read my Bible more and Euclid less.

Robert Buchanan (1841-1901)
(An Old Dominie's Story)

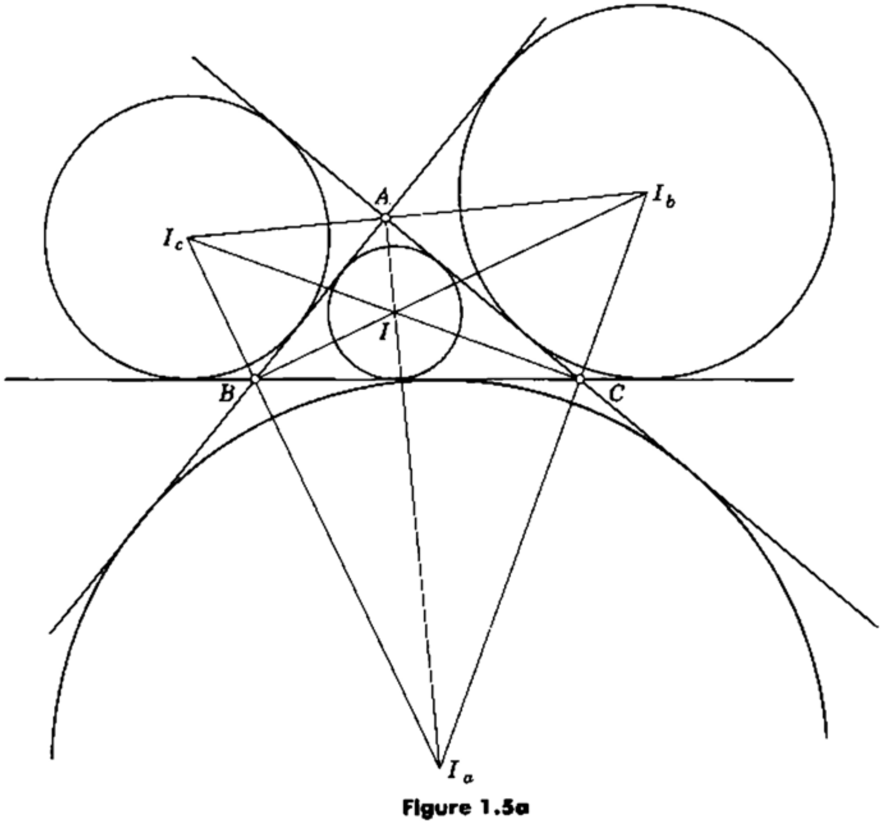


Figure 1.5a

Euclid III.3 tells us that a circle is symmetrical by reflection in any diameter (whereas an ellipse is merely symmetrical about two special diameters: the major and minor axes). It follows that the angle between two intersecting tangents is bisected by the diameter through their common point.

* It is to be understood that any exercise appearing in the form of a theorem is intended to be *proved*. It saves space to omit the words "Prove that" or "Show that."

By considering the loci of points equidistant from pairs of sides of a triangle ABC, we see that the internal and external bisectors of the three angles of the triangle meet by threes in four points I, I_a, I_b, I_c, as in Figure 1.5a. These points are the centers of the four circles that can be drawn to touch the three lines BC, CA, AB. One of them, the *incenter* I, being inside the triangle, is the center of the inscribed circle or *incircle* (Euclid IV.4). The other three are the *excenters* I_a, I_b, I_c: the centers of the three escribed circles or *excircles* [Court **2**, pp. 72–88]. The radii of the incircle and excircles are the *inradius* r and the *exradii* r_a, r_b, r_c.

In describing a triangle ABC, it is customary to call the sides

$$a = BC, \qquad b = CA, \qquad c = AB,$$

the semiperimeter

$$s = \tfrac{1}{2}(a + b + c),$$

the angles A, B, C, and the area Δ.

Since $A + B + C = 180°$, we have

1.51 $$\angle BIC = 90° + \tfrac{1}{2}A,$$

a result which we shall find useful in § 1.9.

Since IBC is a triangle with base a and height r, its area is $\frac{1}{2}ar$. Adding three such triangles we deduce

$$\Delta = \tfrac{1}{2}(a + b + c)r = sr.$$

Similarly $\Delta = \frac{1}{2}(b + c - a)r_a = (s - a)r_a$. Thus

1.52 $$\Delta = sr = (s - a)r_a = (s - b)r_b = (s - c)r_c.$$

From the well-known formula $\cos A = (b^2 + c^2 - a^2)/2bc$, we find also

$$\sin A = [-a^4 - b^4 - c^4 + 2b^2c^2 + 2c^2a^2 + 2a^2b^2]^{\frac{1}{2}}/2bc,$$

whence

1.53 $$\begin{aligned}\Delta &= \tfrac{1}{2}\, bc \sin A \\ &= \tfrac{1}{4}\,[-a^4 - b^4 - c^4 + 2b^2c^2 + 2c^2a^2 + 2a^2b^2]^{\frac{1}{2}} \\ &= \tfrac{1}{4}\,[(a + b + c)(-a + b + c)(a - b + c)(a + b - c)]^{\frac{1}{2}} \\ &= [s(s - a)(s - b)(s - c)]^{\frac{1}{2}}.\end{aligned}$$

This remarkable expression, which we shall use in § 18.4, is attributed to Heron of Alexandria (about 60 A.D.), but it was really discovered by Archimedes. (See B. L. van der Waerden, *Science Awakening*, Oxford University Press, New York, 1961, pp. 228, 277.) Combining Heron's formula with 1.52, we obtain

1.531 $$r^2 = \left(\frac{\Delta}{s}\right)^2 = \frac{(s - a)(s - b)(s - c)}{s}, \quad r_a^{\,2} = \left(\frac{\Delta}{s - a}\right)^2 = \frac{s(s - c)(s - b)}{s - a}.$$

Another consequence of the symmetry of a circle is that the perpendicular bisectors of the three sides of a triangle all pass through the *circumcenter* O,

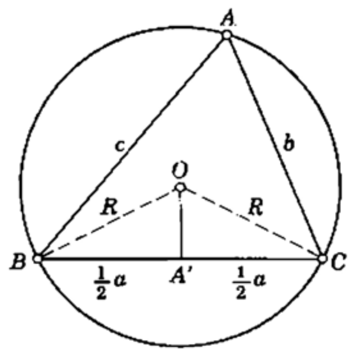


Figure 1.5b

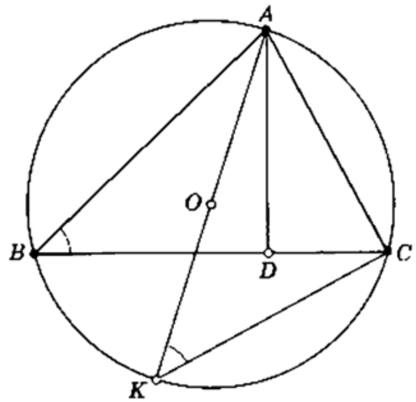


Figure 1.5c

which is the center of the circumscribed circle or *circumcircle* (Euclid IV.5). This is the only circle that can be drawn through the three vertices A, B, C. Its radius R is called the *circumradius* of the triangle. Since the "angle at the center," $\angle BOC$ (Figure 1.5*b*), is double the angle A, the congruent right-angled triangles OBA', OCA' each have an angle A at O, whence

$$R \sin A = BA' = \tfrac{1}{2}a,$$

1.54
$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Draw AD perpendicular to BC, and let AK be the diameter through A of the circumcircle, as in Figure 1.5*c*. By Euclid III.21, the right-angled triangles ABD and AKC are similar; therefore

$$\frac{AD}{AB} = \frac{AC}{AK}, \quad AD = \frac{bc}{2R}.$$

Since $\Delta = \frac{1}{2}BC \times AD$, it follows that

1.55
$$\begin{aligned} 4\Delta R &= abc \\ &= s(s-b)(s-c) + s(s-c)(s-a) + s(s-a)(s-b) \\ &\qquad - (s-a)(s-b)(s-c) \\ &= \frac{\Delta^2}{s-a} + \frac{\Delta^2}{s-b} + \frac{\Delta^2}{s-c} - \frac{\Delta^2}{s} \\ &= \Delta\,(r_a + r_b + r_c - r). \end{aligned}$$

Hence the five radii are connected by the formula

1.56
$$4R = r_a + r_b + r_c - r.$$

Let us now consider four circles E_1, E_2, E_3, E_4, tangent to one another at six distinct points. Each circle E_i has a *bend* ε_i, defined as the reciprocal of its radius with a suitable sign attached, namely, if all the contacts are external (as in the case of the light circles in Figure 1.5*d*), the bends are all positive, but if one circle surrounds the other three (as in the case of the

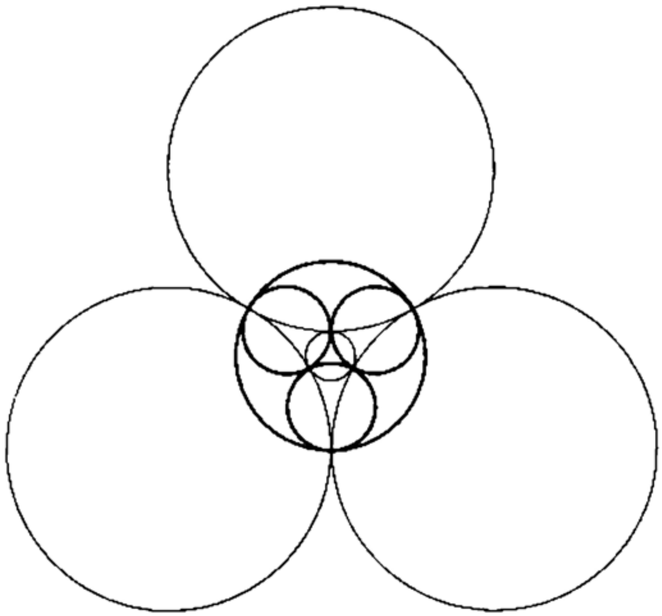


Figure 1.5d

heavy circles) the bend of this largest circle is taken to be negative; and a line counts as a circle of bend 0. In any case, the sum of all four bends is positive.

In a letter of November 1643 to Princess Elisabeth of Bohemia, René Descartes developed a formula relating the radii of four mutually tangent circles. In the "bend" notation it is

1.57 $$2(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2) = (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)^2.$$

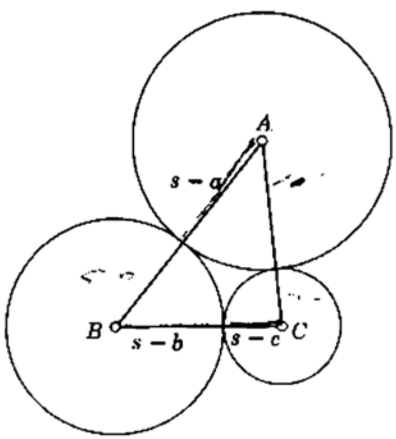


Figure 1.5e

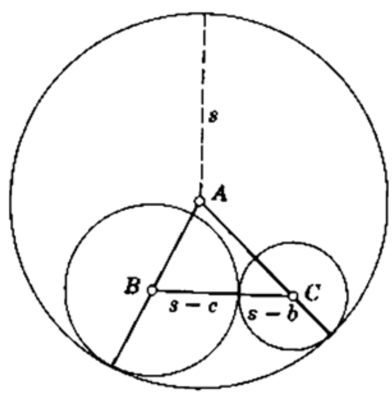


Figure 1.5f

This *Descartes circle theorem* was rediscovered in 1842 by an English amateur, Philip Beecroft, who observed that the four circles E_i determine another set of four circles H_i, mutually tangent at the same six points: H_1 through the three points of contact of E_2, E_3, E_4, and so on. Let η_i denote the bend of H_i. If the centers of E_1, E_2, E_3 form a triangle ABC, H_4 is either the

1.58 $$\varepsilon_1 = \frac{1}{s-a},\ \varepsilon_2 = \frac{1}{s-b},\ \varepsilon_3 = \frac{1}{s-c},\ \eta_4 = \mp\frac{1}{r}.$$

In the latter (Figure 1.5*f*),

$$\varepsilon_1 = -\frac{1}{s},\ \varepsilon_2 = \frac{1}{s-c},\ \varepsilon_3 = \frac{1}{s-b},\ \eta_4 = \pm\frac{1}{r_a}.$$

In either case, we see from 1.531 that

$$\varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1 + \varepsilon_1\varepsilon_2 = \left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} + \frac{1}{\varepsilon_3}\right)\varepsilon_1\varepsilon_2\varepsilon_3 = \eta_4{}^2.$$

Similarly $\eta_2\eta_3 + \eta_3\eta_1 + \eta_1\eta_2 = \varepsilon_4{}^2$, and of course we can permute the subscripts 1, 2, 3, 4. Hence

$$(\Sigma\varepsilon_i)^2 = \varepsilon_1{}^2 + \ldots + \varepsilon_4{}^2 + 2\varepsilon_1\varepsilon_2 + \ldots + 2\varepsilon_3\varepsilon_4 = \Sigma\varepsilon_i{}^2 + \Sigma\eta_i{}^2.$$

Since this expression involves ε_i and η_i symmetrically, it is also equal to $(\Sigma\eta_i)^2$; thus

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 = \eta_1 + \eta_2 + \eta_3 + \eta_4 > 0.$$

Also, since

$$\begin{aligned}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4)(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4) &= (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)^2 - \varepsilon_4{}^2 \\ &= \varepsilon_1{}^2 + \varepsilon_2{}^2 + \varepsilon_3{}^2 - \varepsilon_4{}^2 + 2\eta_4{}^2 \\ &= (\eta_2\eta_3 + \eta_2\eta_4 + \eta_3\eta_4) + (\eta_1\eta_3 + \ldots) + (\eta_1\eta_2 + \ldots) - (\eta_1\eta_2 + \ldots) + 2\eta_4{}^2 \\ &= 2(\eta_1\eta_4 + \eta_2\eta_4 + \eta_3\eta_4) + 2\eta_4{}^2 = 2\eta_4(\eta_1 + \eta_2 + \eta_3 + \eta_4),\end{aligned}$$

1.59 $$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4 = 2\eta_4.$$

Adding four such equations after squaring each side, we deduce $\Sigma\varepsilon_i{}^2 = \Sigma\eta_i{}^2$, whence

$$2\Sigma\varepsilon_i{}^2 = \Sigma\varepsilon_i{}^2 + \Sigma\eta_i{}^2 = (\Sigma\varepsilon_i)^2.$$

Thus 1.57 has been proved.

In 1936, this theorem was rediscovered again by Sir Frederick Soddy, who had received a Nobel prize in 1921 for his discovery of isotopes. He expressed the theorem in the form of a poem, *The Kiss Precise**, of which the middle verse runs as follows:

Four circles to the kissing come.
The smaller are the benter.
The bend is just the inverse of
The distance from the centre.
Though their intrigue left Euclid dumb

* *Nature*, **137** (1936), p. 1021; **139** (1937), p. 62. In the next verse, Soddy announced his discovery of the analogous formula for 5 spheres in 3 dimensions. A final verse, added by Thorold Gosset (1869–1962) deals with $n+2$ spheres in n dimensions; see Coxeter, *Aequationes Mathematicae,* **1** 1968), pp. 104–121.

There's now no need for rule of thumb.
Since zero bend's a dead straight line
And concave bends have minus sign,
The sum of the squares of all four bends
Is half the square of their sum.

EXERCISES

1. Find the locus of the image of a fixed point P by reflection in a variable line through another fixed point O.

2. For the general triangle ABC, establish the identities

$$\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}, \qquad r r_a r_b r_c = \Delta^2.$$

3. The lengths of the tangents from the vertex A to the incircle and to the three excircles are respectively

$$s - a, \quad s, \quad s - c, \quad s - b.$$

4. The circumcenter of an obtuse-angled triangle lies outside the triangle.

5. Where is the circumcenter of a right-angled triangle?

6. Let U, V, W be three points on the respective sides BC, CA, AB of a triangle ABC. The perpendiculars to the sides at these points are concurrent if and only if

$$AW^2 + BU^2 + CV^2 = WB^2 + UC^2 + VA^2.$$

7. A triangle is right-angled if and only if $r + 2R = s$.

8. The bends of Beecroft's eight circles satisfy

$$\varepsilon_1 + \eta_1 = \varepsilon_2 + \eta_2 = \varepsilon_3 + \eta_3 = \varepsilon_4 + \eta_4, \quad \Sigma\varepsilon_i\eta_i = 0.$$

9. For any four numbers satisfying $k + l + m + n = 0$, there is a "Beecroft configuration" having bends

$$\varepsilon_1 = k(k + l), \varepsilon_2 = (k + l)l, \quad \varepsilon_3 = n^2 - kl, \quad \varepsilon_4 = m^2 - kl,$$
$$\eta_1 = l^2 - mn, \eta_2 = k^2 - mn, \eta_3 = m(m + n), \eta_4 = (m + n)n.$$

(*Hint:* Express $\varepsilon_3, \varepsilon_4, \eta_1, \eta_2$ as rational functions of $\varepsilon_1, \varepsilon_2, \eta_3, \eta_4$.)

10. If three circles, externally tangent to one another, have centers forming a triangle ABC, they are all tangent to two other circles (or possibly a circle and a line) whose bends are

$$\frac{r + 4R \pm 2s}{\Delta}.$$

11. Given a point P on the circumcircle of a triangle, the feet of the perpendiculars from P to the three sides all lie on a straight line. (This line is commonly called the *Simson line* of P with respect to the triangle, although it was first mentioned by W. Wallace, thirty years after Simson's death [Johnson **1**, p. 138].)

12. Given a triangle ABC and a point P in its plane (but not on a side nor on the circumcircle), let $A_1B_1C_1$ be the derived triangle formed by the feet of the perpendiculars from P to the sides BC, CA, AB. Let $A_2B_2C_2$ be derived analogously from $A_1B_1C_1$ (using the same P), and $A_3B_3C_3$ from $A_2B_2C_2$. Then $A_3B_3C_3$ is directly similar to ABC. [Casey **1**, p. 253.] (*Hint:* $\angle PBA = \angle PA_1C_1 = \angle PC_2B_2 = \angle PB_3A_3$.) This result has been extended by B. M. Stewart from the third derived triangle of a triangle to the nth derived n-gon of an n-gon. (*American Mathematical Monthly* **47** (1940), pp. 462–466).

1.6 THE EULER LINE AND THE ORTHOCENTER

Although the Greeks worked fruitfully, not only in geometry but also in the most varied fields of mathematics, nevertheless we today have gone beyond them everywhere and certainly also in geometry.

F. Klein (1849-1925)
[Klein **2**, p. 189]

From now on, we shall have various occasions to mention the name of L. Euler (1707–1783), a Swiss who spent most of his life in Russia, making important contributions to all branches of mathematics. Some of his simplest discoveries are of such a nature that one can well imagine the ghost of Euclid saying, "Why on earth didn't I think of that?"

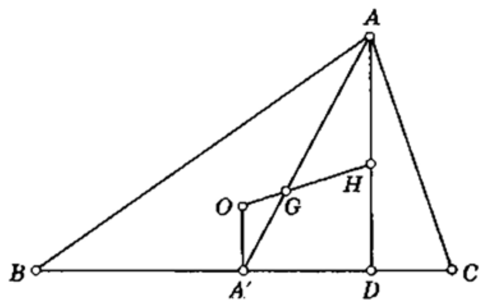


Figure 1.6a

If the circumcenter O and centroid G of a triangle coincide, each median is perpendicular to the side that it bisects, and the triangle is "isosceles three ways," that is, equilateral. Hence, if a triangle ABC is not equilateral, its circumcenter and centroid lie on a unique line OG. On this so-called *Euler line,* consider a point H such that $OH = 3OG$, that is, $GH = 2OG$ (Figure 1.6*a*). Since also $GA = 2A'G$, the latter half of Euclid VI.2 tells us that AH is parallel to $A'O$, which is the perpendicular bisector of BC. Thus AH is perpendicular to BC. Similarly BH is perpendicular to CA, and CH to AB.

The line through a vertex perpendicular to the opposite side is called an *altitude.* The above remarks [cf. Court **2**, p. 101] show that

The three altitudes of any triangle all pass through one point on the Euler line.

This common point H of the three altitudes is called the *orthocenter* of the triangle.

EXERCISES

1. Through each vertex of a given triangle ABC draw a line parallel to the opposite side. The perpendicular bisectors of the sides of the triangle so formed suggest an alternative proof that the three altitudes of ABC are concurrent.

2. The orthocenter of an obtuse-angled triangle lies outside the triangle.

3. Where is the orthocenter of a right-angled triangle?

4. Any triangle having two equal altitudes is isosceles.

5. Construct an isosceles triangle ABC (with base BC), given the median BB' and the altitude BE. (*Hint:* The centroid is two-thirds of the way from B to B'.) (H. Freudenthal.)

6. The altitude AD of any triangle ABC is of length

$$2R \sin B \sin C.$$

7. Find the perpendicular distance from the centroid G to the side BC.

8. If the Euler line passes through a vertex, the triangle is either right-angled or isosceles (or both).

9. If the Euler line is parallel to the side BC, the angles B and C satisfy

$$\tan B \tan C = 3.$$

1.7 THE NINE-POINT CIRCLE

This circle is the first really exciting one to appear in any course on elementary geometry.

Daniel Pedoe (1910-)

[Pedoe **1**, p. 1]

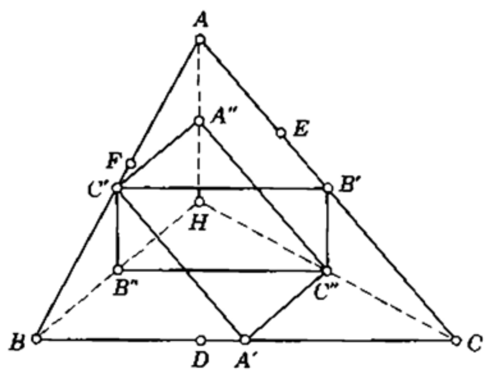


Figure 1.7a

The feet of the altitudes (that is, three points like D in Figure 1.6*a*) form the *orthic triangle* (or "pedal triangle") of ABC. The circumcircle of the orthic triangle is called the *nine-point circle* (or "Feuerbach circle") of the original triangle, because it contains not only the feet of the three altitudes but also six other significant points. In fact,

1.71 *The midpoints of the three sides, the midpoints of the lines joining the orthocenter to the three vertices, and the feet of the three altitudes, all lie on a circle.*

EXERCISES

1. In Figure 1.8*a*, UV and VW make equal angles with CA. Deduce that the orthocenter of any triangle is the incenter of its orthic triangle. (In other words, if ABC is a triangular billiard table, a ball at U, hit in the direction UV, will go round the triangle UVW indefinitely, that is, until it is stopped by friction.)

2. How does Fagnano's problem collapse when we try to apply it to a triangle ABC in which the angle A is obtuse?

3. The circumcircles of the three equilateral triangles in Figure 1.8*c* all pass through P, and their centers form a fourth equilateral triangle.*

4. Three holes, at the vertices of an arbitrary triangle, are drilled through the top of a table. Through each hole a thread is passed with a weight hanging from it below the table. Above, the three threads are all tied together and then released. If the three weights are all equal, where will the knot come to rest?

5. Four villages are situated at the vertices of a square of side one mile. The inhabitants wish to connect the villages with a system of roads, but they have only enough material to make $\sqrt{3} + 1$ miles of road. How do they proceed? [Courant and Robbins **1**, p. 392.]

6. Solve Fermat's problem for a triangle ABC with $A > 120°$, and for a convex quadrangle $ABCD$.

7. If two points P, P', inside a triangle ABC, are so situated that $\angle CBP = \angle PBP' = \angle P'BA$, $\angle ACP' = \angle P'CP = \angle PCB$, then $\angle BP'P = \angle PP'C$.

8. If four squares are placed externally (or internally) on the four sides of any parallelogram, their centers are the vertices of another square. [Yaglom **1**, pp. 96–97.]

9. Let X, Y, Z be the centers of squares placed externally on the sides BC, CA, AB of a triangle ABC. Then the segment AX is congruent and perpendicular to YZ (also BY to ZX and CZ to XY). (W. A. J. Luxemburg.)

10. Let Z, X, U, V be the centers of squares placed externally on the sides AB, BC, CD, DA of any simple quadrangle (or "quadrilateral") $ABCD$. Then the segment ZU (joining the centers of two "opposite" squares) is congruent and perpendicular to XV. [Forder **2**, p. 40.]

1.9 MORLEY'S THEOREM

Many of the proofs in mathematics are very long and intricate. Others, though not long, are very ingeniously constructed.

E. C. Titchmarsh (1899–1963)

[Titchmarsh **1**, p. 23]

One of the most surprising theorems in elementary geometry was discovered about 1899 by F. Morley (whose son Christopher wrote novels such as *Thunder on the Left*). He mentioned it to his friends, who spread it over

* Court [**1**, pp. 105–107]. See also *Mathesis* 1938, p. 293 (footnote, where this theorem is attributed to Napoleon); and Forder [**2**, p. 40] for some interesting generalizations.

Extend the sides of the isosceles triangles below their bases until they meet again in points A, B, C. Since $\alpha + \beta + \gamma + 60° = 180°$, we can immediately infer the measurement of some other angles, as marked in Figure 1.9a. For instance, the triangle AQR must have an angle $60° - \alpha$ at its vertex A, since its angles at Q and R are $\alpha + \beta$ and $\gamma + \alpha$.

Referring to 1.51, we see that one way to characterize the incenter I of a triangle ABC is to describe it as lying on the bisector of the angle A at such a distance that

$$\angle BIC = 90° + \tfrac{1}{2}A.$$

Applying this principle to the point P in the triangle $P'BC$, we observe that the line PP' (which is a median of both the equilateral triangle PQR and the isosceles triangle $P'QR$) bisects the angle at P'. Also the half angle at P' is $90° - \alpha$, and

$$\angle BPC = 180° - \alpha = 90° + (90° - \alpha).$$

Hence P is the incenter of the triangle $P'BC$. Likewise Q is the incenter of $Q'CA$, and R of $R'AB$. Therefore all the three small angles at C are equal; likewise at A and at B. In other words, the angles of the triangle ABC are trisected.

The three small angles at A are each $\frac{1}{3}A = 60° - \alpha$; similarly at B and C. Thus

$$\alpha = 60° - \tfrac{1}{3}A, \quad \beta = 60° - \tfrac{1}{3}B, \quad \gamma = 60° - \tfrac{1}{3}C.$$

By choosing these values for the base angles of our isosceles triangles, we can ensure that the above procedure yields a triangle ABC that is similar to any given triangle.

This completes the proof.

EXERCISES

1. The three lines PP', QQ', RR' (Figure 1.9a) are concurrent. In other words, the trisectors of A, B, C meet again to form another triangle $P'Q'R'$ which is perspective with the equilateral triangle PQR. (In general $P'Q'R'$ is *not* equilateral.)

2. What values of α, β, γ will make the triangle ABC (i) equilateral, (ii) right-angled isosceles? Sketch the figure in each case.

3. Let P_1 and P_2 (on CA and AB) be the images of P by reflection in CP' and BP'. Then the four points P_1, Q, R, P_2 are evenly spaced along a circle through A. In the special case when the triangle ABC is equilateral, these four points occur among the vertices of a regular enneagon (9-gon) in which A is the vertex opposite to the side QR.

2
Regular Polygons

We begin this chapter by discussing (without proofs) the possibility of constructing certain regular polygons with the instruments allowed by Euclid. We then consider all these polygons, regardless of the question of constructibility, from the standpoint of symmetry. Finally, we extend the concept of a regular polygon so as to include star polygons.

2.1 CYCLOTOMY

One, two! One, two! And through and through
The vorpal blade went snicker-snack!

Lewis Carroll
[Dodgson **2**, Chap. 1]

Euclid's postulates imply a restriction on the instruments that he allowed for making constructions, namely the restriction to ruler (or straightedge) and compasses. He constructed an equilateral triangle (I.1), a square (IV.6), a regular pentagon (IV.11), a regular hexagon (IV.15), and a regular 15-gon (IV.16). The number of sides may be doubled again and again by repeated angle bisections. It is natural to ask which other regular polygons can be constructed with Euclid's instruments. This question was completely answered by Gauss (1777–1855) at the age of nineteen [see Smith **2**, pp. 301–302]. Gauss found that a regular n-gon, say $\{n\}$, can be so constructed if the odd prime factors of n are distinct "Fermat primes"

$$F_k = 2^{2^k} + 1.$$

The only known primes of this kind are

$$F_0 = 2^1 + 1 = 3, \quad F_1 = 2^2 + 1 = 5, \quad F_2 = 2^4 + 1 = 17,$$
$$F_3 = 2^8 + 1 = 257, \quad F_4 = 2^{16} + 1 = 65537.$$

2.3 ISOMETRY

One way of describing the structure of space, preferred by both Newton and Helmholtz, is through the notion of congruence. Congruent parts of space V, V′ are such as can be occupied by the same rigid body in two of its positions. If you move the body from one into the other position the particle of the body covering a point P of V will afterwards cover a certain point P′ of V′, and thus the result of the motion is a mapping P → P′ of V upon V′. We can extend the rigid body either actually or in imagination so as to cover an arbitrarily given point P of space, and hence the congruent mapping P → P′ can be extended to the entire space.

Hermann Weyl (1885-1955)

[Weyl **1**, p. 43]

We shall find it convenient to use the word *transformation* in the special sense of a one-to-one correspondence $P \to P'$ among all the points in the plane (or in space), that is, a rule for associating pairs of points, with the understanding that each pair has a first member P and a second member P' and that every point occurs as the first member of just one pair and also as the second member of just one pair. It may happen that the members of a pair coincide, that is, that P' coincides with P; in this case P is called an *invariant* point (or "double point") of the transformation.

In particular, an *isometry* (or "congruent transformation," or "congruence") is a transformation which preserves length, so that, if (P, P') and (Q, Q') are two pairs of corresponding points, we have $PQ = P'Q'$: PQ and $P'Q'$ are congruent segments. For instance, a *rotation* of the plane about P (or about a line through P perpendicular to the plane) is an isometry having P as an invariant point, but a *translation* (or "parallel displacement") has no invariant point: every point is moved.

A *reflection* is the special kind of isometry in which the invariant points consist of all the points on a line (or plane) called the *mirror.*

A still simpler kind of transformation (so simple that it may at first seem too trivial to be worth mentioning) is the *identity,* which leaves every point unchanged. The result of applying several transformations successively is called their *product.* If the product of two transformations is the identity, each is called the *inverse* of the other, and their product in the reverse order is again the identity.

2.31 *If an isometry has more than one invariant point, it must be either the identity or a reflection.*

To prove this, let A and B be two invariant points, and P any point not on the line AB (Figure 1.3*b*). The corresponding point P', satisfying

$$AP' = AP, \qquad BP' = BP,$$

must lie on the circle with center A and radius AP, and on the circle with cen-

we say that S is of *period* 4. Similarly S^2, being a half-turn, is of period 2 [see Coxeter **1**, p. 39]. The only transformation of period 1 is the identity. A translation is aperiodic (that is, it has no period), but it is conveniently said to be of infinite period.

Some figures admit both reflections and rotations as symmetry operations. The letter H (Figure 2.4*d*) has a horizontal mirror (like E) and a vertical mirror (like A), as well as a center of rotational symmetry (like N) where the two mirrors intersect. Thus it has four symmetry operations: the identity 1, the horizontal reflection R_1, the vertical reflection R_2, and the half-turn $R_1R_2 = R_2R_1$.

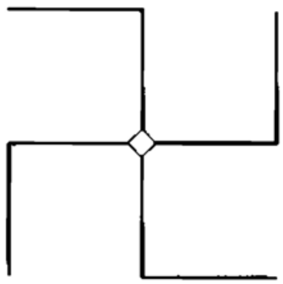


Figure 2.4c

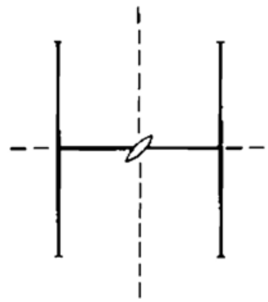


Figure 2.4d

EXERCISES

1. Every isometry of period 2 is either a reflection or a half-turn [Bachmann **1**, pp. 2–3].

2. Express (*a*) a half-turn, (*b*) a quarter-turn, as transformations of (i) Cartesian coordinates, (ii) polar coordinates. (Take the origin to be the center of rotation.)

2.5 GROUPS

Symmetry, as wide or as narrow as you may define its meaning, is one idea by which man through the ages has tried to comprehend and create order, beauty, and perfection.

Hermann Weyl [**1**, p. 5]

A set of transformations [Birkhoff and MacLane **1**, pp. 115–118] is said to form a *group* if it contains the inverse of each and the product of any two (including the product of one with itself or with its inverse). The number of distinct transformations is called the *order* of the group. (This may be either finite or infinite.) Clearly the symmetry operations of any figure form a group. This is called the *symmetry group* of the figure. In the extreme case when the figure is completely irregular (like the numeral 6) its symmetry group is of order one, consisting of the identity alone.

This is the same as R_1R_2 if the two mirrors are at right angles, in which case R_1R_2 is a half-turn and $(R_1R_2)^2 = 1$.

EXERCISES

1. The product of quarter-turns (in the same sense) about C and B is the half-turn about the center of a square having BC for a side.

2. Let $ACPQ$ and $BARS$ be squares on the sides AC and BA of a triangle ABC. If B and C remain fixed while A varies freely, PS passes through a fixed point.

2.7 THE KALEIDOSCOPE

D_2 is a special case of the general dihedral group D_n, which is, for $n > 2$, the symmetry group of the regular n-gon, $\{n\}$. (See Figure 2.7*a* for the cases $n = 3, 4, 5$.) This is evidently a group of order $2n$, consisting of n rotations (through the n effectively distinct multiples of $360°/n$) and n reflections. When n is odd, each of the n mirrors joins a vertex to the midpoint of the opposite side; when n is even, $\frac{1}{2}n$ mirrors join pairs of opposite vertices and $\frac{1}{2}n$ bisect pairs of opposite sides [see Birkhoff and MacLane **1**, pp. 117–118, 135].

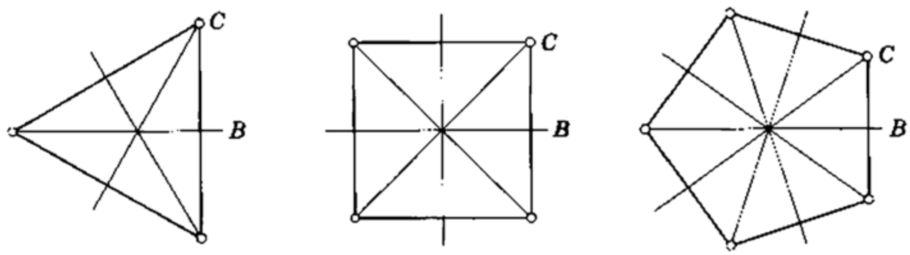


Figure 2.7a

The n rotations are just the operations of the cyclic group C_n. Thus the operations of D_n include all the operations of C_n: in technical language, C_n is a *subgroup* of D_n. The rotation through $360°/n$, which generates the subgroup, may be described as the product $S = R_1R_2$ of reflections in two adjacent mirrors (such as OB and OC in Figure 2.7*a*) which are inclined at $180°/n$.

Let $R_1, R_2, \ldots, R_n$ denote the n reflections in their natural order of arrangement. Then R_1R_{k+1}, being the product of reflections in two mirrors inclined at k times $180°/n$, is a rotation through k times $360°/n$:

$$R_1R_{k+1} = S^k.$$

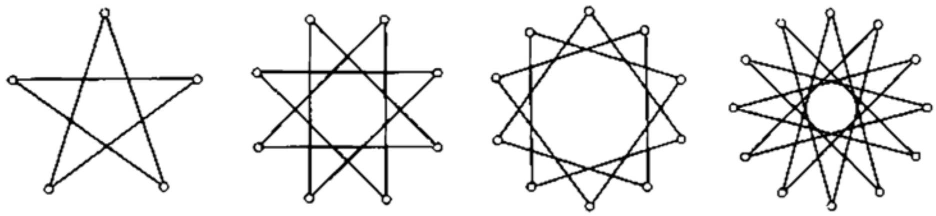
Thus $R_{k+1} = R_1S^k$, and the n reflections may be expressed as

$$R_1, R_1S, R_1S^2, \ldots, R_1S^{n-1}.$$

In other words, D_n is generated by R_1 and S. By substituting R_1R_2 for S, we

ordinary regular p-gon, $\{p\}$. When $d > 1$, the sides cross one another, but the crossing points are not counted as vertices. Since d may be any positive integer relatively prime to p and less than $\frac{1}{2}p$, there is a regular polygon $\{n\}$ for each rational number $n > 2$. In fact, it is occasionally desirable to include also the *digon* $\{2\}$, although its two sides coincide.

When $p = 5$, we have the pentagon $\{5\}$ of density 1 and the *pentagram* $\{\frac{5}{2}\}$ of density 2, which was used as a special symbol by the Babylonians and by the Pythagoreans. Similarly, the *octagram* $\{\frac{8}{3}\}$ and the *decagram* $\{\frac{10}{3}\}$ have density 3, while the *dodecagram* $\{\frac{12}{5}\}$ has density 5 (Figure 2.8*a*). These particular polygons have names as well as symbols because they occur as faces of interesting polyhedra and tessellations.*



$\left\{\frac{5}{2}\right\}$ $\left\{\frac{8}{3}\right\}$ $\left\{\frac{10}{3}\right\}$ $\left\{\frac{12}{5}\right\}$

Figure 2.8a

Polygons for which $d > 1$ are known as *star* polygons. They are frequently used in decoration. The earliest mathematical discussion of them was by Thomas Bradwardine (1290–1349), who became archbishop of Canterbury for the last month of his life. They were also studied by the great German scientist Kepler (1571–1630) [see Coxeter **1**, p. 114]. It was the Swiss mathematician L. Schläfli (1814–1895) who first used a numerical symbol such as $\{p/d\}$. This notation is justified by the occurrence of formulas that hold for $\{n\}$ equally well whether n be an integer or a fraction. For instance, any side of $\{n\}$ forms with the center O an isosceles triangle OP_0P_1 (Figure 2.8*b*) whose angle at O is $2\pi/n$. (As we are introducing trigonometrical ideas, it is natural to use radian measure and write 2π instead of 360°.) The base of this isosceles triangle, being a side of the polygon, is conveniently denoted by $2l$. The other sides of the triangle are equal to the circumradius R of the polygon. The altitude or median from O is the inradius r of the polygon. Hence

2.81 $$R = l \csc \frac{\pi}{n}, \qquad r = l \cot \frac{\pi}{n}.$$

If $n = p/d$, the area of the polygon is naturally defined to be the sum of the areas of the p isosceles triangles, namely

* H. S. M. Coxeter, M. S. Longuet-Higgins, and J. C. P. Miller, Uniform polyhedra, *Philosophical Transactions of the Royal Society*, **A, 246** (1954), pp. 401–450.

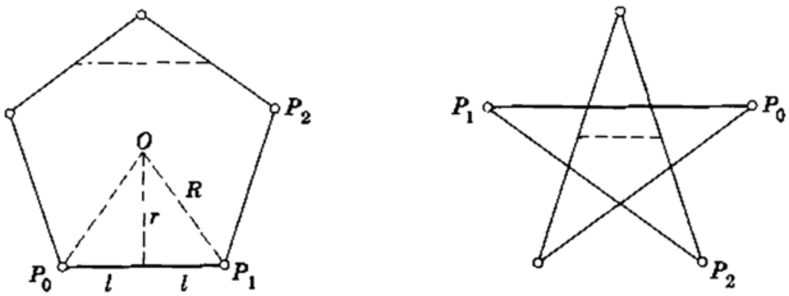


Figure 2.8b

2.82
$$plr = pl^2 \cot \frac{\pi}{n}.$$

When $d = 1$, this is simply $pl^2 \cot \pi/p$; in other cases our definition of area has the effect that every part of the interior is counted a number of times equal to the "local density" of that part; for example, the pentagonal region in the middle of the pentagram $\{\frac{5}{2}\}$ is counted twice.

The angle $P_0P_1P_2$ between two adjacent sides of $\{n\}$, being the sum of the base angles of the isosceles triangle, is the supplement of $2\pi/n$, namely

2.83
$$\left(1 - \frac{2}{n}\right)\pi.$$

The line segment joining the midpoints of two adjacent sides is called the *vertex figure* of $\{n\}$. Its length is clearly

2.84
$$2l \cos \frac{\pi}{n}$$

[Coxeter **1**, pp. 16, 94].

EXERCISES

1. If the sides of a polygon inscribed in a circle are all equal, the polygon is regular.

2. If a polygon inscribed in a circle has an odd number of vertices, and all its angles are equal, the polygon is regular. (Marcel Riesz.)

3. Find the angles of the polygons

$$\{5\},\quad \{\tfrac{5}{2}\},\quad \{9\},\quad \{\tfrac{9}{2}\},\quad \{\tfrac{9}{4}\}.$$

4. Find the radii and vertex figures of the polygons

$$\{8\},\quad \{\tfrac{8}{3}\},\quad \{12\},\quad \{\tfrac{12}{5}\}.$$

5. Give polar coordinates for the kth vertex P_k of a polygon $\{n\}$ of circumradius 1 with its center at the pole, taking P_0 to be $(1, 0)$.

6. Can a square cake be cut into nine slices so that everyone gets the same amount of cake and the same amount of icing?

3

Isometry in the Euclidean plane

Having made some use of reflections, rotations, and translations, we naturally ask why a rotation or a translation can be achieved as a continuous displacement (or "motion") while a reflection cannot. It is also reasonable to ask whether there is any other kind of isometry that resembles a reflection in this respect. After answering these questions in terms of "sense," we shall use the information to prove a remarkable theorem (§ 3.6) and to describe the seven possible ways to repeat a pattern on an endless strip (§ 3.7).

3.1 DIRECT AND OPPOSITE ISOMETRIES

"Take care of the sense, and the sounds will take care of themselves."

Lewis Carroll

[Dodgson **1**, Chap. 9]

By several applications of Axiom 1.26, it can be proved that any point P in the plane of two congruent triangles ABC, $A'B'C'$ determines a corresponding point P' such that $AP = A'P'$, $BP = B'P'$, $CP = C'P'$. Likewise another point Q yields Q', and $PQ = P'Q'$. Hence

3.11 *Any two congruent triangles are related by a unique isometry.*

In § 1.3, we saw that Pappus's proof of *Pons asinorum* involved the comparison of two coincident triangles ABC, ACB. We see intuitively that this is a distinction of *sense*: if one is counterclockwise the other is clockwise. It is a "topological" property of the Euclidean plane that this distinction can be extended from coincident triangles to distinct triangles: any two "directed" triangles, ABC and $A'B'C'$, either agree or disagree in sense. (For a deeper investigation of this intuitive idea, see Veblen and Young [**2**, pp. 61–62] or Denk and Hofmann [**1**, p. 56].)

If ABC and $A'B'C'$ are congruent, the isometry that relates them is said to be *direct* or *opposite* according as it preserves or reverses sense, that is,

3.13 *Every isometry of the plane is the product of at most three reflections. If there is an invariant point, "three" can be replaced by "two."*

We prove this in four stages, using 3.11. Trivially, if the triangles ABC, $A'B'C'$ coincide, the isometry is the identity (which is the product of a reflection with itself). If A coincides with A', and B with B', while C and C' are distinct, the triangles are related by the reflection in AB. The case when only A coincides with A' can be reduced to one of the previous cases by reflecting ABC in m, the perpendicular bisector of BB' (see Figure 3.1*b*). Finally, the general case can be reduced to one of the first three cases by reflecting ABC in the perpendicular bisector of AA' [Coxeter **1**, p. 35].

Since a reflection reverses sense, an isometry is direct or opposite according as it is the product of an even or odd number of reflections.

Since the identity is the product of two reflections (namely of any reflection with itself), we may say simply that any isometry is the product of *two* or *three* reflections, according as it is direct or opposite. In particular,

3.14 *Any isometry with an invariant point is a rotation or a reflection according as it is direct or opposite.*

EXERCISES

1. Name two direct isometries.
2. Name one opposite isometry. Is there any other kind?
3. If AB and $A'B'$ are related by a rotation, how can the center of rotation be constructed? (*Hint*: The perpendicular bisectors of AA' and BB' are not necessarily distinct.)
4. The product of reflections in three lines through a point is the reflection in another line through the same point [Bachmann **1**, p. 5].

3.2 TRANSLATION

Enoch walked with God; and he was not, for God took him.

Genesis V, 24

The particular isometries so far considered, namely reflections (which are opposite) and rotations (which are direct), have each at least one invariant point. A familiar isometry that leaves no point invariant is a *translation* [Bachmann **1**, p. 7], which may be described as the product of half-turns about two distinct points O, O' (Figure 3.2*a*). The first half-turn transforms an arbitrary point P into P^{H}, and the second transforms this into P^{T}, with the final result that *PP^{T} is parallel to OO' and twice as long.* Thus the length and direction of PP^{T} are constant: independent of the position of P. Since a translation is completely determined by its length and direction, the product of half-turns about O and O' is the same as the product of half-turns about Q and Q', provided QQ' is equal and parallel to OO'. (This

means that $OO'Q'Q$ is a parallelogram, possibly collapsing to form four collinear points, as in Figure 3.2*a*.) Thus, for a given translation, the center of one of the two half-turns may be arbitrarily assigned.

3.21 *The product of two translations is a translation.*

For, we may arrange the centers so that the first translation is the product of half-turns about O_1 and O_2, while the second is the product of half-turns about O_2 and O_3. When they are combined, the two half-turns about O_2 cancel, and we are left with the product of half-turns about O_1 and O_3.

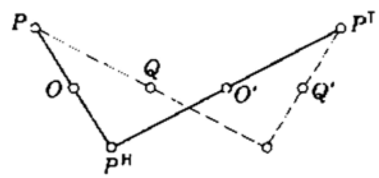


Figure 3.2a

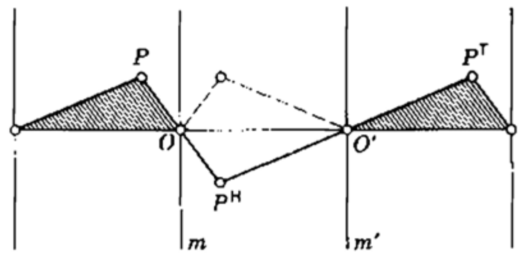


Figure 3.2b

Similarly, if m and m' (Figure 3.2*b*) are the lines through O and O' perpendicular to OO', the half-turns about O and O' are the products of reflections in m and OO', OO' and m'. When they are combined, the two reflections in OO' cancel, and we are left with the product of reflections in m and m'. Hence

3.22 *The product of reflections in two parallel mirrors is a translation through twice the distance between the mirrors.*

If a translation T takes P to P^{T} and Q to Q^{T}, the segment QQ^{T} is equal and parallel to PP^{T}; therefore $PQQ^{\mathrm{T}}P^{\mathrm{T}}$ is a parallelogram. Similarly, if another translation U takes P to Q, it also takes P^{T} to Q^{T}; therefore

$$\mathrm{TU} = \mathrm{UT}.$$

(In detail, if Q is P^{U}, Q^{T} is P^{UT}. But U takes P^{T} to P^{TU}. Therefore P^{TU} and P^{UT} coincide, for all positions of P.) In other words,

3.23 *Translations are commutative.*

The product of a half-turn H and a translation T is another half-turn; for we can express the translation as the product of two half-turns, one of which is H, say $\mathrm{T} = \mathrm{HH}'$, and then we have

$$\mathrm{HT} = \mathrm{H}^2\mathrm{H}' = \mathrm{H}':$$

3.24 *The product of a half-turn and a translation is a half-turn.*

EXERCISES

1. If T is the product of half-turns about O and O', what is the product of half-turns about O' and O?

2. When a translation is expressed as the product of two reflections, to what extent can one of the two mirrors be arbitrarily assigned?

We have seen (Figure 3.2b) that the product of reflections in two parallel mirrors m, m' is a translation. This may be regarded as the limiting case of a rotation whose center is very far away; for the two parallel mirrors are the limiting case of two mirrors intersecting at a very small angle. Accordingly, the infinite group generated by a single translation is denoted by C_∞, and the infinite group generated by two parallel reflections is denoted by D_∞. Abstractly, C_∞ is the "free group with one generator." If T is the generating translation, the group consists of the translations

$$\ldots, \mathrm{T}^{-2}, \quad \mathrm{T}^{-1}, \quad 1, \quad \mathrm{T}, \quad \mathrm{T}^2, \ldots.$$

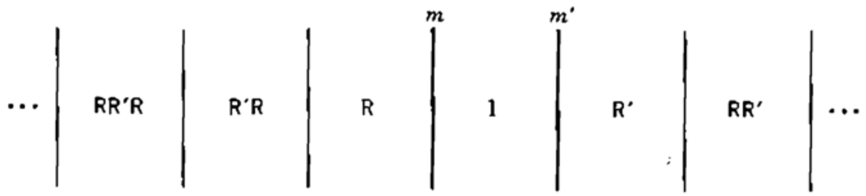


Figure 3.7b

Similarly, D_∞, generated by the reflections R, R′ in parallel mirrors m, m' (Figure 3.7b), consists of the reflections and translations

$$\ldots, \mathrm{RR'R}, \quad \mathrm{R'R}, \quad \mathrm{R}, \quad 1, \quad \mathrm{R'}, \quad \mathrm{RR'}, \quad \mathrm{R'RR'}, \ldots$$

[Coxeter **1**, p. 76]; its abstract definition is simply

$$\mathrm{R}^2 = \mathrm{R'}^2 = 1.$$

This group can be observed when we sit in a barber's chair between two parallel mirrors (cf. the *New Yorker*, Feb. 23, 1957, p. 39, where somehow the reflection RR′RR′R yields a demon).

A different geometrical representation for the same abstract group D_∞ is obtained by interpreting the generators R and R′ as half-turns. There is also an intermediate representation in which one of them is a reflection and the other a half-turn; but in this case their product is no longer a translation but a glide reflection.

Continuing in this manner, we could soon obtain the complete list of the seven infinite "one-dimensional" symmetry groups: the seven essentially distinct ways to repeat a pattern on a strip or ribbon [Speiser **1**, pp. 81–82]:

Typical pattern	*Generators*	*Abstract Group*
(i) . . . L L L L . . .	1 translation	C_∞
(ii) . . . L Γ L Γ . . .	1 glide reflection	
(iii) . . . V V V V . . .	2 reflections	D_∞
(iv) . . . N N N N . . .	2 half-turns	
(v) . . . V Λ V Λ . . .	1 reflection and 1 half-turn	
(vi) . . . D D D D . . .	1 translation and 1 reflection	$C_\infty \times D_1$
(vii) . . . H H H H . . .	3 reflections	$D_\infty \times D_1$

terstices (Figure 4.1*c*). (We use the term *tessellation* for any arrangement of polygons fitting together so as to cover the whole plane without overlapping.)

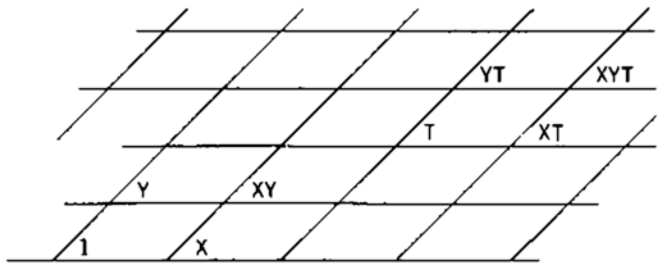


Figure 4.1c

A typical parallelogram is formed by the four points 1, X, XY, Y. The translation $T = X^x Y^y$ transforms this parallelogram into another one having the point T (instead of 1) at its "first" corner. There is thus a one-to-one correspondence between the cells or tiles of the tessellation and the transformations in the group, with the property that each transformation takes any point inside the original cell to a point similarly situated in the new cell. For this reason, the typical parallelogram is called a *fundamental region*.

The shape of the fundamental region is far from unique. Any parallelogram will serve, provided it has four lattice points for its vertices but no others on its boundary or inside [Hardy and Wright **1**, p. 28]. This is the geometrical counterpart of the algebraic statement that the group generated by X, Y is equally well generated by $X^a Y^b$, $X^c Y^d$, provided

$$ad - bc = \pm 1.$$

To express the old generators in terms of the new, we observe that

$$(X^a Y^b)^d (X^c Y^d)^{-b} = X^{ad-bc}, \quad (X^a Y^b)^{-c} (X^c Y^d)^a = Y^{ad-bc}.$$

But there is no need for the fundamental region to be a parallelogram at all; for example, we may replace each pair of opposite sides by a pair of congruent curves, as in Figure 4.1*d*.

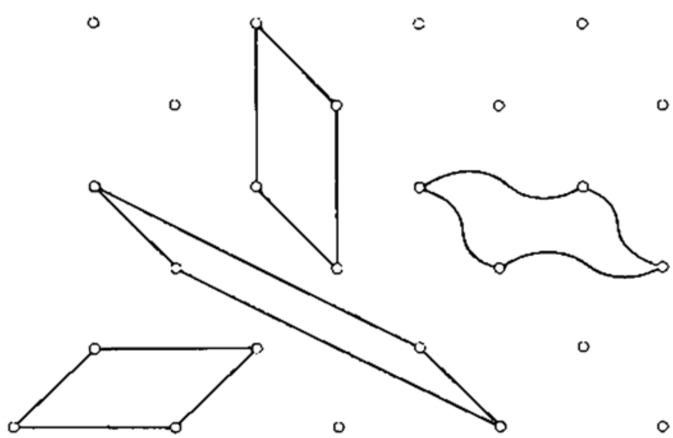


Figure 4.1d

EXERCISES

1. Why do the vertices of the quadrangles in Figure 4.2*c* form two superposed la
:es?

2. Draw the tessellation of Dirichlet regions for a given lattice. Divide each regi
:o two halves by means of a diagonal. The resulting tessellation is a special case of t
;sellation of scalene triangles (Figure 4.2*b*) or of irregular quadrangles (Figure 4.2
cording as the Dirichlet region is rectangular or hexagonal.

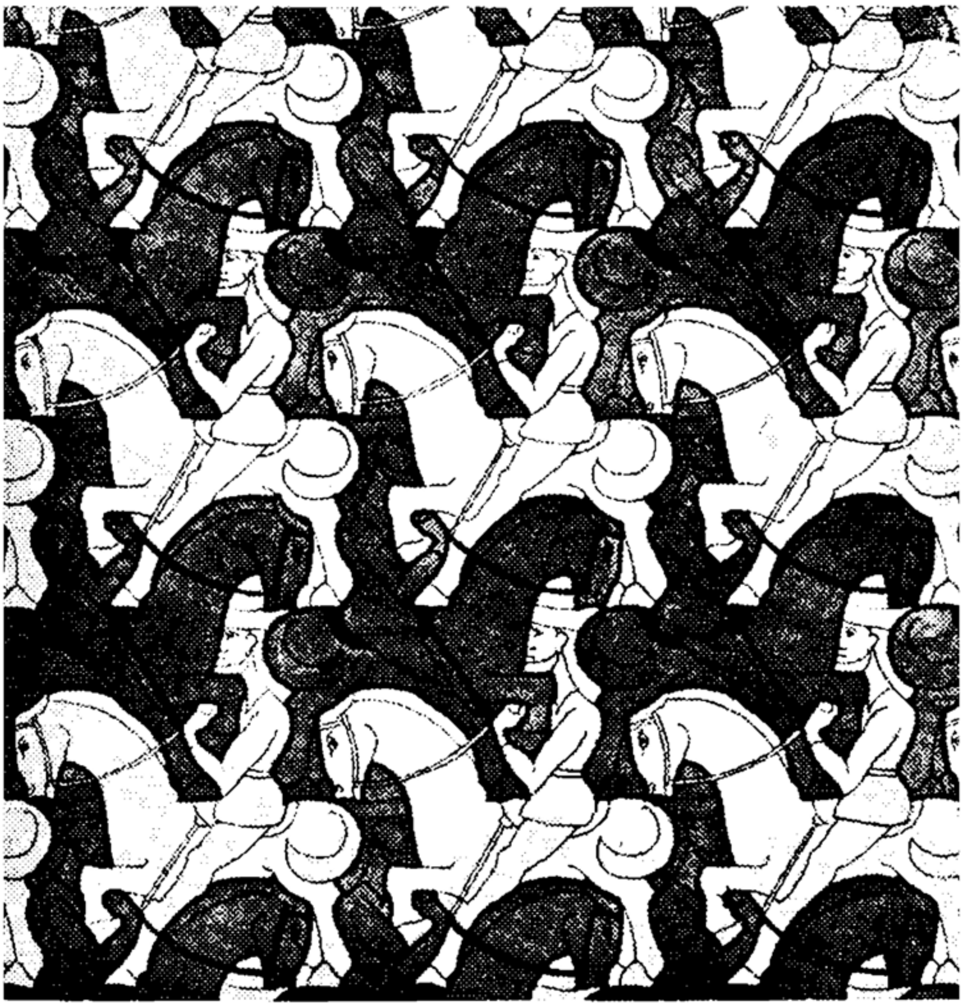


Plate I

.3 THE ART OF M. C. ESCHER

The groups **p1** and **p2** are two of the simplest of the seventeen discre
roups of isometries involving two independent translations. Several othe
ill be mentioned in this section and the next. Convenient generators for
f them are listed in Table I on p. 413.

The art of filling a plane with a repeating pattern reached its highest development in thirteenth-century Spain, where the Moors used all the seventeen groups in their intricate decoration of the Alhambra [Jones **1**]. Their preference for abstract patterns was due to their strict observance of the Second Commandment. The Dutch artist M. C. Escher, free from such scruples, makes an ingenious application of these groups by using animal shapes for their fundamental regions. For instance, the symmetry group of his pattern of knights on horseback (Plate I) seems at first sight to be **p1**, generated by a horizontal translation and a vertical translation. But by ignoring the distinction between the dark and light specimens we obtain the more interesting group **pg**, which is generated by two parallel glide reflections, say G and G′. We observe that the vertical translation can be expressed equally well as G^2 or G'^2. It is remarkable that the single relation

$$G^2 = G'^2$$

provides a complete abstract definition for this group [Coxeter and Moser **1**, p. 43]. Clearly, the knight and his steed (of either color) constitute a fundamental region for **pg**. But we must combine two such regions, one dark and one light, in order to obtain a fundamental region for **p1**.

Similarly, the symmetry group of Escher's pattern of beetles (Plate II) seems at first sight to be **pm**, generated by two vertical reflections and a vertical translation. But on looking more closely we see that there are both dark and light beetles, and that the colors are again interchanged by glide reflections. The complete symmetry group **cm**, whose fundamental region is the right or left half of a beetle of either color, is generated by any such vertical glide reflection along with a vertical reflection. To obtain a fundamental region for the "smaller" group **pm**, we combine the right half of a beetle of either color with the left half of an adjacent beetle of the other color.

A whole beetle (of either color) provides a fundamental region for the group **p1** (with one of its generating translations oblique) or equally well for **pg**.

EXERCISES

1. Locate the axes of two glide reflections which generate **pg** in Plates I and II.
2. Any two parallelograms whose sides are in the same two directions can together be repeated by translations to fill the plane.

4.4 SIX PATTERNS OF BRICKS

Figure 4.4*a* shows how six of the seventeen two-dimensional space groups arise as the symmetry groups of familiar patterns of rectangles, which we may think of as bricks or tiles. The generators are indicated as follows: a

Similarly, it transforms any point Q on BB' into a point Q' on BB'. If AA' and BB' are not parallel, these two invariant lines intersect in an invariant point O. Hence

5.13 *Any dilatation that is not a translation has an invariant point.*

This invariant point O is *unique.* For, a dilatation that has two invariant points O_1 and O_2 can only be the identity, which may reasonably be regarded as a translation, namely a translation through distance zero [Weyl **1**, p. 69].

Clearly, any point P is transformed into a point P' on OP. Let us write

$$OP' = \lambda OP,$$

with the convention that the number λ is positive or negative according as P and P' are on the same side of O or on opposite sides. With the help of some homothetic triangles (as in Figure 5.1*b*), we see that λ is a constant, that is, independent of the position of P. Moreover, any segment PQ is transformed into a segment $|\lambda|$ times as long, and oppositely directed if $\lambda < 0$. We shall use the symbol $O(\lambda)$ for the dilatation with center O and ratio λ. (Court [**2**, p. 40] prefers "(O, λ)".)

In particular, $O(1)$ is the identity and $O(-1)$ is a half-turn. Clearly, the only dilatations which are also isometries are half-turns and translations. In the case of a translation, such a symbol as $O(\lambda)$ is no longer available.

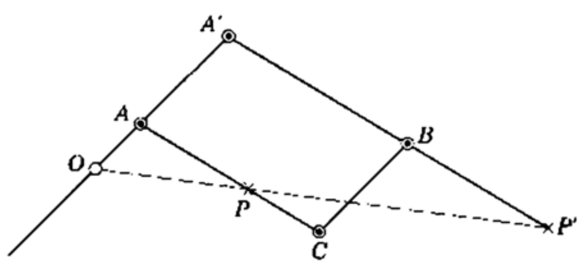


Figure 5.1c

EXERCISES

1. What is the inverse of the dilatation $O(\lambda)$?

2. If the product $O_1(\lambda_1)$ and $O_2(\lambda_2)$ is $O(\lambda_1\lambda_2)$, where is O?

3. Express the dilatation $O(\lambda)$ in terms of (a) polar coordinates, (b) Cartesian coordinates.

4. Explain the action of the *pantograph* (Figure 5.1*c*), an instrument invented by Christoph Scheiner about 1630 for the purpose of making a copy, reduced or enlarged, of any given figure. It is formed by four rods, hinged at the corners of a parallelogram $AA'BC$ whose angles are allowed to vary. The three collinear points O, P, P', on the respective rods AA', AC, $A'B$, remain collinear when the shape of the parallelogram is changed. The instrument is pivoted at O. When a pencil point is inserted at

5.4 THE INVARIANT POINT OF A SIMILARITY

When a figure is enlarged so as to remain still of the same shape, every straight line in it remains a straight line, and every angle remains congruent to itself. All the parts of the figure are equally enlarged. When one figure is an enlarged copy of another, the two are said to be similar. The degree of enlargement necessary to make one figure equal to the other is called their ratio of similitude. The ratio of two lines in the one figure is equal to the ratio of the two corresponding lines in the other.

W. K. Clifford (1845-1879)
(*Mathematical Papers*, p. 631)

A *similarity* (or "similarity transformation," or "similitude") is a transformation which takes each segment AB into a segment $A'B'$ whose length is given by

$$\frac{A'B'}{AB} = \mu,$$

where μ is a constant positive number (the same for all segments) called the *ratio of magnification* (Clifford's "ratio of similitude"). It follows that any triangle is transformed into a similar triangle, and any angle into an equal (or opposite) angle. When $\mu = 1$, the similarity is an isometry. Other special cases are the dilatations $O(\pm\mu)$.

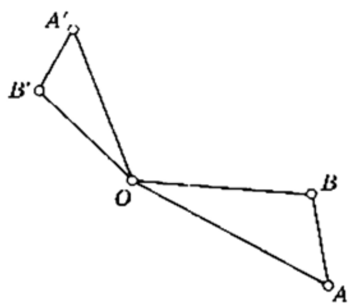


Figure 5.4a

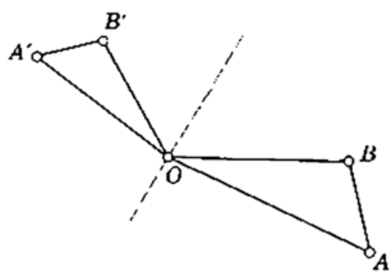


Figure 5.4b

A less familiar instance is the *dilative rotation* (or "spiral similarity", Figure 5.4a), which is the product of a dilatation $O(\mu)$ and a rotation about O. Another is the *dilative reflection* (Figure 5.4b), which is the product of a dilatation $O(\mu)$ and the reflection in a line through O. We would not obtain anything new (in either case) if we replaced this dilatation $O(\mu)$ by $O(-\mu)$. For, since $O(-\mu) = O(-1) \cdot O(\mu)$, and $O(-1)$ is a half-turn, the product of $O(\mu)$ and a rotation through α about O is the same as the product of $O(-\mu)$ and a rotation through $\alpha + \pi$; and since $O(-1)$ is the product of two perpendicular reflections, the product of $O(\mu)$ and the reflection in a line m

of circles (infinitely many), and they are all orthogonal to the circle of inversion. Hence

6.21 *The inverse of a given point P is the second intersection of any two circles through P orthogonal to the circle of inversion.*

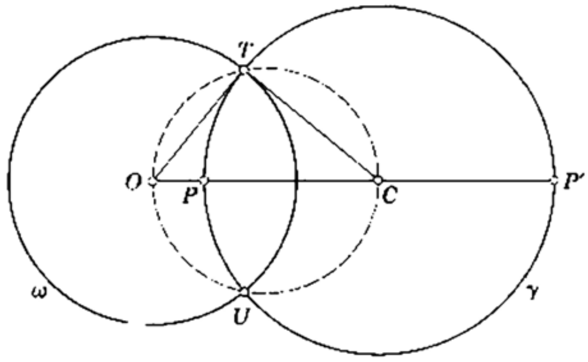


Figure 6.2a

The above remarks provide a simple solution for the problem of drawing, through a given point P, a circle (or line) orthogonal to two given circles. Let P_1, P_2 be the inverses of P in the two circles. Then the circle PP_1P_2 (or the line through these three points, if they happen to be collinear) is orthogonal to the two given circles.

If O and C are the centers of two orthogonal circles ω and γ, as in Figure 6.2*a*, the circle on OC as diameter passes through the points of intersection T, U. Every other point on this circle is inside one of the two orthogonal circles and outside the other. It follows that, if a and b are two perpendicular lines through O and C respectively, either a touches γ and b touches ω, or a cuts γ and b lies outside ω, or a lies outside γ and b cuts ω.

6.3 INVERSION OF LINES AND CIRCLES

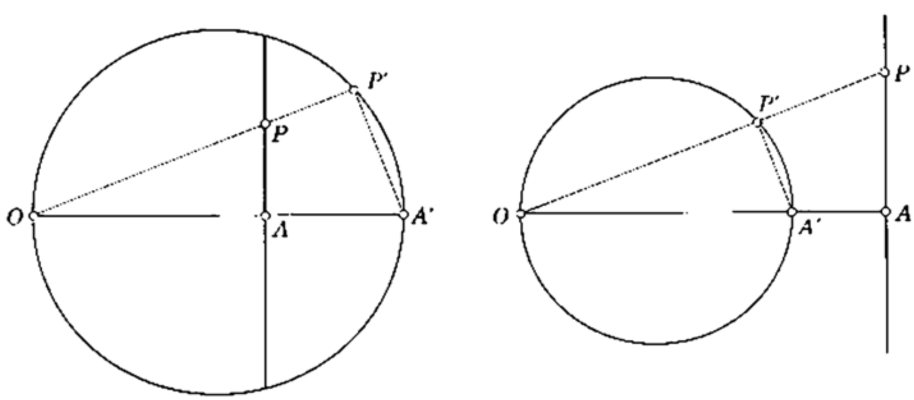


Figure 6.3a

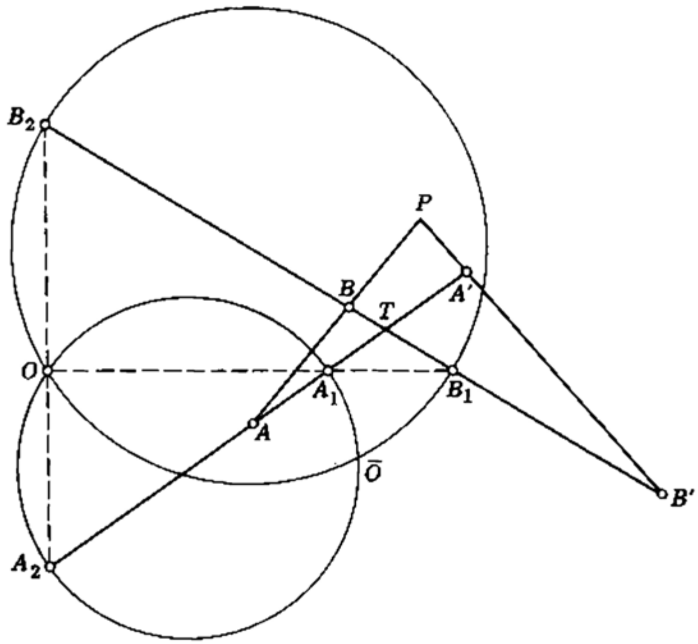


Figure 6.6b

5. Let the *inversive distance* between two nonintersecting circles be defined as the natural logarithm of the ratio of the radii (the larger to the smaller) of two concentric circles into which the given circles can be inverted. Then, if a nonintersecting pencil of coaxal circles includes α_1, α_2, α_3 (in this order), the three inversive distances satisfy

$$(\alpha_1, \alpha_2) + (\alpha_2, \alpha_3) = (\alpha_1, \alpha_3).$$

6. Two given unequal circles are related by infinitely many dilative rotations and by infinitely many dilative reflections. The locus of invariant points (in either case) is the circle having for diameter the segment joining the two centers of similitude of the given circles. (This locus is known as the *circle of similitude* of the given circles.) What is the corresponding result for two given *equal* circles?

7. The inverses, in two given circles, of a point on their circle of similitude, are images of each other by reflection in the radical axis of the two circles [Court **2**, p. 199].

6.7 CIRCLE-PRESERVING TRANSFORMATIONS

Having observed that inversion is a transformation of the whole inversive plane (including the point at infinity) into itself, taking circles into circles, we naturally ask what is the most general transformation of this kind. We distinguish two cases, according as the point at infinity is, or is not, invariant.

In the former case, not only are circles transformed into circles but also lines into lines. With the help of Euclid III.21 (see p. 7) we deduce that equality of angles is preserved, and consequently the measurement of angles is preserved, so that every triangle is transformed into a similar triangle, and the transformation is a similarity (§ 5.4).

If, on the other hand, the given transformation T takes an ordinary point

O into the point at infinity O', we consider the product J_1T, where J_1 is the inversion in the unit circle with center O. This product J_1T, leaving O' invariant, is a similarity. Let k^2 be its ratio of magnification, and J_k the inversion in the circle with center O and radius k. Since, by 6.11, J_1J_k is the dilatation $O(k^2)$, the similarity J_1T can be expressed as J_1J_kS, where S is an isometry. Thus

$$T = J_kS,$$

the product of an inversion and an isometry.

To sum up,

6.71 *Every circle-preserving transformation of the inversive plane is either a similarity or the product of an inversion and an isometry.*

It follows that every circle-preserving transformation is the product of at most four inversions (provided we regard a reflection as a special kind of inversion) [Ford **1**, p. 26]. Such a transformation is called a *homography* or an *antihomography* according as the number of inversions is even or odd. The product of two inversions (either of which could be just a reflection) is called a *rotary* or *parabolic* or *dilative* homography according as the two inverting circles are intersecting, tangent, or nonintersecting (i.e., according as the orthogonal pencil of invariant circles is nonintersecting, tangent, or intersecting). As special cases we have, respectively, a rotation, a translation, and a dilatation. The most important kind of rotary homography is the *Möbius involution,* which, being the inversive counterpart of a half-turn, is the product of inversions in two orthogonal circles (e.g., the product of the inversion in a circle and the reflection in a diameter). Any product of four inversions that cannot be reduced to a product of two is called a *loxodromic* homography [Ford **1**, p. 20].

EXERCISE

When a given circle-preserving transformation is expressed as JS (where J is an inversion and S an isometry), J and S are unique. There is an equally valid expression SJ′, in which the isometry precedes the inversion. Why does this revised product involve the same S? Under what circumstances will we have $J' = J$?

6.8 INVERSION IN A SPHERE

By revolving Figures 6.1*a*, 6.2*a*, 6.3*a*, 6.3*b*, and 6.4*a* about the line of centers (OP or OA or OC), we see that the whole theory of inversion extends readily from circles in the plane to spheres in space. Given a sphere with center O and radius k, we define the inverse of any point P (distinct from O) to be the point P' on the ray OP whose distance from O satisfies

$$OP \times OP' = k^2.$$

Alternatively, P' is the second intersection of three spheres through P orthogonal to the sphere of inversion. Every sphere inverts into a sphere, ro-

7

Isometry and similarity in Euclidean space

This chapter is the three-dimensional counterpart of Chapters 3 and 5. In § 7.5 we find a proof (independent of Euclid's Fifth Postulate) for the theorem (discovered by Michel Chasles in 1830) that every motion is a *twist*. In § 7.6 we see that every similarity (except the twist and the glide reflection, which are isometries) is a three-dimensional *dilative rotation*.

Most isometries are familiar in everyday life. When you walk straight forward you are undergoing a translation. When you turn a corner, it is a rotation; when you ascend a spiral staircase, a twist. The transformation that interchanges yourself and your image in an ordinary mirror is a reflection, and it is easy to see how you could combine this with a rotation or a translation to obtain a rotatory reflection or a glide reflection, respectively.

7.1 DIRECT AND OPPOSITE ISOMETRIES

> *A congruence is either proper, carrying a left screw into a left and a right one into a right, or it is improper or reflexive, changing a left screw into a right one and vice versa. The proper congruences are those transformations which . . . connect the positions of points of a rigid body before and after a motion.*
>
> H. Weyl [**1**, pp. 43-44]

The axioms of congruence, a sample of which was given in 1.26, can be extended in a natural manner from plane geometry to solid geometry. In space, an *isometry* (Weyl's "congruence") is still any transformation that preserves length, so that a line segment PQ is transformed into a congruent seg-

the central inversion $O(-1)$ is opposite, as we have seen.

In space, as in the plane, two similar figures are related by a *similarity*, which in special cases may be an isometry or a dilatation. By a natural extension of the terminology we now take a *dilative rotation* to mean the product of a rotation about a line l (the *axis*) and a dilatation whose center O lies on l. The plane through O perpendicular to l is invariant, being transformed according to the two-dimensional "dilative rotation" of § 5.5. In the special case when the rotation about l is a half-turn, there are infinitely many other invariant planes, namely all the planes through l. Any such plane is transformed according to a dilative reflection.

Suppose a dilative rotation is the product of a rotation through angle α and a dilatation $O(\lambda)$ (where O lies on the axis). The following values of α and λ yield special cases which are familiar:

α	λ	Similarity
0	1	Identity
π	1	Half-turn
α	1	Rotation
π	-1	Reflection
0	-1	Central inversion
α	-1	Rotatory inversion
0	λ	Dilatation

We observe that this table includes all kinds of isometry, both direct and opposite, except the translation, twist and glide reflection (which have no invariant points). Still more surprisingly, we shall find that, with these same three exceptions, *every similarity is a dilative rotation.*

The role of similar triangles is now taken over by similar tetrahedra. Evidently

7.61 *Two given similar tetrahedra $ABCD$, $A'B'C'D'$ are related by a unique similarity $ABCD \rightarrow A'B'C'D'$, which is direct or opposite according as the sense of $A'B'C'D'$ agrees or disagrees with that of $ABCD$.*

In other words, a similarity is completely determined by its effect on any four given non-coplanar points, and we have the following generalization of Theorem 7.13:

7.62 *Two given similar triangles ABC, $A'B'C'$ are related by just two similarities: one direct and one opposite.*

As a step towards proving that every similarity which is not an isometry is a dilative rotation, let us first prove

7.63 *Every similarity which is not an isometry has just one invariant point.*

Part II

8
Coordinates

In the preceding chapters, a few exercises on coordinates have been inserted for the sake of those readers who are already aquainted with analytic geometry. Other readers, having omitted such exercises, are awaiting enlightenment at the present stage. In addition to the usual rectangular Cartesian coordinates, we shall consider oblique and polar coordinates. (The polar equation for an ellipse is important because of its use in the theory of orbits.) After a brief mention of special curves we shall give an outline of Newton's application of calculus to problems of arc length and area. The section on three-dimensional space culminates in a surprising property of the doughnut-shaped *torus*.

8.1 CARTESIAN COORDINATES

Though the idea behind it all is childishly simple, yet the method of analytic geometry is so powerful that very ordinary boys of seventeen can use it to prove results which would have baffled the greatest of the Greek geometers—Euclid, Archimedes, and Apollonius.

E. T. Bell (1883-1960)

[E. T. Bell **1**, *p.* 21]

Analytic geometry may be described as the representation of the points in n-dimensional space by ordered sets of n (or more) numbers called *coordinates*. For instance, any position on the earth can be specified by its latitude, longitude, and height above sea level.

The one-dimensional case is well illustrated by a thermometer. There is a certain point on the line associated with the number 0; the positive integers 1, 2, 3, . . . are evenly spaced in one direction away from 0, the negative integers -1, -2, -3, . . . in the opposite direction, and the fractional numbers are interpolated in the natural manner. The *displacement* from one point x to another point x' is the positive or negative number $x' - x$.

In the two-dimensional case, the position of a point in a plane may be specified by its distances from two fixed perpendicular lines, the *axes*. This notion can be traced back over two thousand years to Archimedes of Syracuse and Apollonius of Perga, or even to the ancient Egyptians; but it was first developed systematically by two Frenchmen: Pierre Fermat (whose problem about a triangle we solved in § 1.8) and René Descartes (1596–1650). In their formulation the two distances were taken to be positive or zero. The important idea of allowing one or both to be negative was supplied by Sir Isaac Newton (1642–1727), and it was G. W. Leibniz (1646–1716) who first called them "coordinates." (The Germans write *Koordinaten*, the French *coordonnées*.)

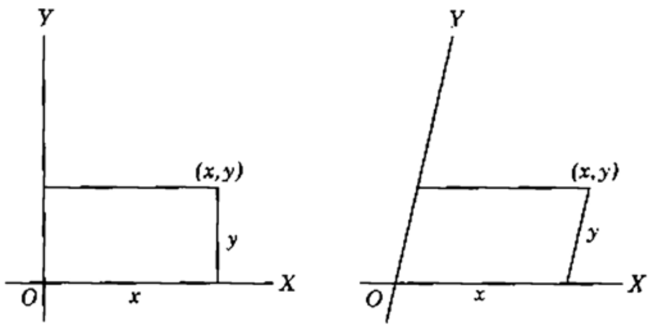


Figure 8.1a

For some purposes it is just as easy to use oblique axes, as in the second part of Figure 8.1*a*. Starting from the *origin* O, where the axes intersect, we reach the general point (x, y) by going a distance x along the x-axis OX and then a distance y along a line parallel to the y-axis OY. The x-axis is said to have the *equation* $y = 0$ because every point $(x, 0)$ satisfies this equation; similarly, $x = 0$ is the equation of the y-axis. On any other line through the origin, consideration of homothetic triangles shows that the ratio y/x is constant; thus any line through the origin $(0, 0)$ may be expressed as $ax + by = 0$.

To obtain the equation for any other line, we take a point (x_1, y_1) on it. In terms of new coordinates x', y', derived by translating the origin from $(0, 0)$ to (x_1, y_1), the line may be expressed as $ax' + by' = 0$. Since $x' = x - x_1$ and $y' = y - y_1$, the same line, in terms of the original coordinates, is

$$a(x - x_1) + b(y - y_1) = 0$$

or, say,

8.11 $$ax + by + c = 0.$$

Thus every line has a *linear* equation, and every linear equation determines a line. In particular, the line that makes intercepts p and q on the axes is

8.12
$$\frac{x}{p} + \frac{y}{q} = 1;$$

for, this equation is linear and is satisfied by both $(p, 0)$ and $(0, q)$. Two lines of the form 8.11 are parallel if they have the same ratio a/b (including, as one possibility, $b = 0$ for both, in which case they are parallel to the y-axis). The point of intersection of two nonparallel lines is obtained by solving the two simultaneous equations for x and y.

If $b \neq 0$, the equation 8.11 may be solved for y in the form $y = -(ax + c)/b$. More generally, points whose coordinates satisfy an equation $F(x, y) = 0$ or $y = f(x)$ can be plotted by giving convenient values to the *abscissa* x and calculating the corresponding values of the *ordinate* y. This procedure is particularly appropriate when $f(x)$ is a one-valued function of x. In other cases we may prefer to use *parametric* equations, expressing x and y as functions of a single variable (or *parameter*) t. For instance, if P_1 denotes the point (x_1, y_1), any line through P_1 has parametric equations

8.13
$$x = x_1 + Xt, \quad y = y_1 + Yt,$$

where X and Y depend on the *direction* of the line.

Sometimes, for the sake of symmetry, the single parameter t is replaced by two parameters, t_1 and t_2, related by an auxiliary equation. For instance, the general point (x, y) on the line through two given points P_1 and P_2 is given by

$$x = t_1x_1 + t_2x_2, \quad y = t_1y_1 + t_2y_2, \quad t_1 + t_2 = 1.$$

This point P, dividing the segment P_1P_2 in the ratio $t_2 : t_1$, is the *centroid* (or "center of gravity") of masses t_1 at P_1 and t_2 at P_2. Positions outside the interval from P_1 (where $t_2 = 0$) to P_2 (where $t_1 = 0$) are covered by allowing t_2 or t_1 to be negative, while still satisfying $t_1 + t_2 = 1$; we may justify this by calling them "electric charges" instead of "masses."

For problems involving the distance between two points or the angle between two lines, it is often advisable to use *rectangular* axes, so that the distance from the origin to (x, y) is the square root of $x^2 + y^2$, and the distance P_1P_2 is the square root of

$$(x_1 - x_2)^2 + (y_1 - y_2)^2.$$

Multiplication of the expression $l = ax + by + c$ by a suitable number enables us to *normalize* the equation $l = 0$ of the general line so that $a^2 + b^2 = 1$. Writing $l = 0$ in the form

$$(x - x_1 + 2al_1)^2 + (y - y_1 + 2bl_1)^2 = (x - x_1)^2 + (y - y_1)^2,$$

where $l_1 = ax_1 + by_1 + c$, we recognize it as the locus of points equidistant from

$$(x_1 - 2al_1, y_1 - 2bl_1) \quad \text{and} \quad (x_1, y_1);$$

in other words, the line $l = 0$ serves as a mirror which interchanges these two points by reflection. It follows that the foot of the perpendicular from P_1 to $l = 0$ is $(x_1 - al_1, y_1 - bl_1)$, and that the distance from P_1 to the line is $\pm l_1$ (provided $a^2 + b^2 = 1$). In particular, the distance from the origin to $l = 0$ is $\pm c$.

The locus of points at unit distance from the origin is the circle

$$x^2 + y^2 = 1,$$

which has the parametric equations

$$x = \cos\theta, \qquad y = \sin\theta$$

or, with $t = \tan\frac{1}{2}\theta$,

$$x = \frac{1 - t^2}{1 + t^2}, \qquad y = \frac{2t}{1 + t^2}.$$

EXERCISES

1. In terms of general Cartesian coordinates, the point (x, y) will be transformed into

$(-x, -y)$ by the half-turn $O(-1)$ (§ 5.1),
$(\mu x, \mu y)$ by the dilatation $O(\mu)$,
$(x + a, y)$ by a translation along the x-axis.

2. In terms of rectangular Cartesian coordinates, the point (x, y) will be transformed into

$(x, -y)$ by reflection in the x-axis,
(y, x) by reflection in the line $x = y$,
$(-y, x)$ by a quarter-turn about the origin,
$(x + a, -y)$ by a glide reflection (in and along the x-axis),
$(\mu x, -\mu y)$ by a dilative reflection (§ 5.6).

3. Let M_{ij} denote the midpoint of P_iP_j. For any four points P_1, P_2, P_3, P_4, the midpoints of $M_{12}M_{34}$, $M_{13}M_{24}$, $M_{14}M_{23}$ all coincide.

8.2 POLAR COORDINATES

The deriving of short cuts from basic principles covers some of the finest achievements of the greatest mathematicians.

M. H. A. Newman (1897-)
(*Mathematical Gazette* **43** (1959), p. 170)

For problems involving directions from a fixed origin (or "pole") O, we often find it convenient to specify a point P by its *polar coordinates* (r, θ), where r is the distance OP and θ is the angle that the direction OP makes with a given *initial line* OX, which may be identified with the x-axis of rectangular Cartesian coordinates. Of course, the point (r, θ) is the same as $(r, \theta + 2n\pi)$ for any integer n. It is sometimes desirable to allow r to be negative, so that (r, θ) is the same as $(-r, \theta + \pi)$.

Given the Cartesian equation for a curve, we can deduce the polar equation for the same curve by substituting

8.21 $$x = r\cos\theta, \qquad y = r\sin\theta.$$

For instance, the unit circle $x^2 + y^2 = 1$ has the polar equation

$$(r\cos\theta)^2 + (r\sin\theta)^2 = 1,$$

which reduces to

$$r = 1.$$

(The positive value of r is sufficient if we allow θ to take all values from $-\pi$ to π or from 0 to 2π.) This procedure is helpful in elementary trigonometry, where students often experience some difficulty in proving (and remembering) the trigonometrical functions of obtuse and larger angles. Taking an angle XOP with $OP = 1$, we can simply *define* its cosine and sine to be the abscissa and ordinate of P.

Polar coordinates are particularly suitable for describing those isometries (§ 3.5) and similarities (§ 5.4) which have an invariant point; for this point may be used as the origin. Thus the general point (r, θ) will be transformed into

$(r, \theta + \alpha)$	by a rotation through α,
$(r, \theta + \pi)$	by a half-turn,
$(r, -\theta)$	by reflection in the initial line,
$(r, 2\alpha - \theta)$	by reflection in the line $\theta = \alpha$,
$(\mu r, \theta)$	by the dilatation $O(\mu)$,
$(\mu r, \theta + \alpha)$	by a dilative rotation with center O,
$(\mu r, 2\alpha - \theta)$	by a dilative reflection with center O and axis $\theta = \alpha$.

Likewise, inversion in the circle $r = k$ (see § 6.1) will transform (r, θ) into

$$(k^2/r, \quad \theta).$$

The Cartesian expressions for the same transformations can be deduced at once. For instance, the rotation through α about O transforms (x, y) into (x', y') where, by 8.21,

$$\begin{aligned} x' &= r\cos(\theta + \alpha) = r(\cos\theta\cos\alpha - \sin\theta\sin\alpha) = x\cos\alpha - y\sin\alpha, \\ y' &= r\sin(\theta + \alpha) = r(\cos\theta\sin\alpha + \sin\theta\cos\alpha) = x\sin\alpha + y\cos\alpha. \end{aligned}$$

In particular, a quarter-turn transforms (x, y) into $(-y, x)$, and it follows that a necessary and sufficient condition for two points (x, y) and (x', y') to lie in perpendicular directions from the origin is

8.22 $$xx' + yy' = 0.$$

Such a transformation as

8.23 $$\begin{aligned} x' &= x\cos\alpha - y\sin\alpha, \\ y' &= x\sin\alpha + y\cos\alpha \end{aligned}$$

has two distinct aspects: an "active" or *alibi* aspect, in which each point

Conversely, any two circles that satisfy this relation are orthogonal. In particular, the circles

8.32 $$x^2 + y^2 + 2gx + c = 0,$$

8.33 $$x^2 + y^2 + 2fy - c = 0,$$

whose centers lie on the x- and y-axes respectively, are orthogonal. Keeping c constant and allowing g or f to take various values, we obtain two orthogonal pencils of coaxal circles, whose radical axes are $x = 0$ and $y = 0$ respectively. If $c = 0$, we have two orthogonal *tangent* pencils, each consisting of all the circles that touch one of the axes at the origin. If $c > 0$, the circles 8.32, for various values of g, form a *nonintersecting* pencil, including the two point circles

$$(x + g)^2 + y^2 = 0, \qquad g = \pm\sqrt{c},$$

which are the limiting points $(\mp\sqrt{c}, 0)$ of the pencil. The circles 8.33, which pass through these two points, form the orthogonal *intersecting* pencil.

EXERCISES

1. The circle $x^2 + y^2 = k^2$ inverts (x, y) into

$$\left(\frac{k^2\,x}{x^2 + y^2}, \quad \frac{k^2\,y}{x^2 + y^2}\right)$$

Apply this inversion to the line 8.11 and to the circle 8.31.

2. Find the locus of a point (x, y) whose distances from $(k/\mu, 0)$ and $(\mu k, 0)$ are in the ratio $1 : \mu$ (cf. § 6.6).

3. Obtain the Cartesian equation of the locus of a point the product of whose distances from $(a, 0)$ and $(-a, 0)$ is a^2. Deduce the polar equation of this "figure of eight," which is the *lemniscate of Jacob Bernoulli*.

4. Given two equal circles in contact, find the locus of the vertices of triangles for which the first is the nine-point circle (§ 1.7), the second is an excircle (§ 1.5). (*Answer:* A lemniscate.*)

5. A circle of radius b rolls without sliding on the outside of a fixed circle of radius nb. The locus of a point fixed on the circumference of the rolling circle is called an *epicycloid* (when n is an integer, an *n-cusped* epicycloid). Obtain the parametric equations

8.34 $$x = (n + 1)b \cos t - b \cos (n + 1)t,$$
$$y = (n + 1)b \sin t - b \sin (n + 1)t.$$

Sketch the cases $n = 1$ (the *cardioid*), $n = 2$ (the *nephroid*), $n = 3$, and $n = \frac{3}{2}$. [See Robson **1**, p. 368.]

6. Shifting the origin to the cusp $(b, 0)$, obtain the polar equation

$$r = 2b\,(1 - \cos\theta)$$

for the cardioid (8.34 with $n = 1$). Deduce that chords through the cusp are of constant length.

* Richard Blum, *Canadian Mathematical Bulletin*, **1** (1958), pp. 1–3.

7. A circle of radius b rolls without sliding on the inside of a fixed circle of radius nb, where $n > 1$. Find parametric equations for the *hypocycloid* (when n is integral, the *n-cusped* hypocycloid) which is the locus of a point fixed on the circumference of the rolling circle. Sketch the cases $n = 2$ (which is surprising), $n = 3$ (the *deltoid*), and $n = 4$ (the *astroid*). Eliminate the parameter in the last two cases, obtaining, for the astroid,

$$x^{2/3} + y^{2/3} = a^{2/3} \qquad (a = 4b)$$

[Lamb **2**, pp. 297–303].

Steiner discovered that all the Simson lines for any given triangle touch a deltoid. Three of the lines, namely, those parallel to the sides of Morley's equilateral triangle (§ 1.9), are the "apsidal" tangents which the deltoid shares with the nine-point circle. Their points of contact are the vertices of the equilateral triangle XYZ described in Ex. 3 on page 20. For details, see Baker [**1**, pp. 330–349, especially p. 347].

8.4 CONICS

In addition to the straight lines, circles, planes and spheres with which every student of Euclid is familiar, the Greeks knew the properties of the curves given by cutting a cone with a plane—the ellipse, parabola and hyperbola. Kepler discovered by analysis of astronomical observations, and Newton proved mathematically on the basis of the inverse square law of gravitational attraction, that the planets move in ellipses. The geometry of ancient Greece thus became the cornerstone of modern astronomy.

J. L. Synge (1897 -)
[Synge **2**, p. 32]

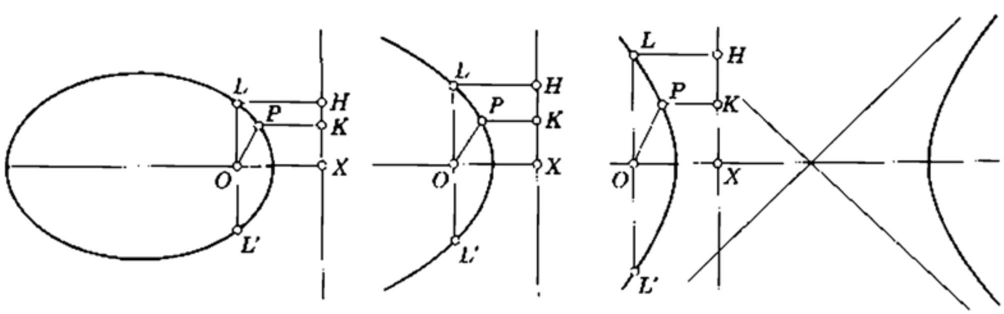


Figure 8.4a

There are several different ways to define a conic (or "conic section"). One of the most straightforward is the following (cf. § 6.6): A *conic* is the

locus of a point P whose distance OP from a fixed point O is ε times its distance PK from a fixed line HX (Figure 8.4a), where ε is a positive constant.

Other definitions for a conic, proposed by Menaechmus about 340 B.C., were reconciled with this one by Pappus of Alexandria (fourth century A.D.) or possibly by Euclid [see Coolidge **1**, pp. 9–13].

The conic is called an *ellipse* if $\varepsilon < 1$, a *parabola* if $\varepsilon = 1$, a *hyperbola* if $\varepsilon > 1$. (These names are due to Apollonius.)

The point O and the line HX are called a *focus* and the corresponding *directrix*. The number ε, called the *eccentricity*, is usually denoted by e (but then, to avoid any possible misunderstanding, we should add "where e need not be the base of the natural logarithms" [Littlewood **1**, p. 43]). The chord LL' through the focus, parallel to the directrix, is called the *latus rectum;* its length is denoted by $2l$, so that

$$l = OL = \varepsilon LH.$$

In terms of polar coordinates with the initial line OX perpendicular to the directrix, we have

$$r = OP = \varepsilon PK = \varepsilon(LH - r\cos\theta)$$

8.41
$$= l - \varepsilon r \cos\theta,$$

so that

8.42
$$\frac{l}{r} = 1 + \varepsilon\cos\theta.$$

Since this equation is unchanged when we replace θ by $-\theta$, the conic is symmetrical by reflection in the initial line. When $\theta = 0$, $r = l/(1 + \varepsilon)$; and when $\theta = \pi$, $r = l/(1 - \varepsilon)$; therefore the conic meets the initial line twice except when $\varepsilon = 1$.

If $\varepsilon < 1$, 8.42 makes r finite and positive for all values of θ; therefore the ellipse is a *closed* (oval) curve. If $\varepsilon = 1$, r is still finite and positive except when $\theta = \pi$; therefore the parabola is not closed but extends to infinity in one direction. If $\varepsilon > 1$, r is positive or negative according as $\cos\theta$ is greater or less than $-1/\varepsilon$; therefore the hyperbola consists of two separate branches, given by

$$-\alpha < \theta < \alpha, \qquad \alpha < \theta < 2\pi - \alpha,$$

respectively, where $\alpha = \operatorname{arcsec}(-\varepsilon)$.

Squaring 8.41, we obtain the Cartesian equation

8.43
$$x^2 + y^2 = (l - \varepsilon x)^2$$

(which indicates that a circle may be regarded as an ellipse with eccentricity zero). If $\varepsilon \neq 1$, we can divide by $1 - \varepsilon^2$ and then write a for $l/(1 - \varepsilon^2)$, obtaining

$$x^2 + \frac{y^2}{1 - \varepsilon^2} = la - 2\varepsilon ax$$

These lines are called the *asymptotes* of the hyperbola. If $a = b$, they are perpendicular, and we have a *rectangular* (or "equilateral") hyperbola.

If $\varepsilon = 1$, then 8.43 reduces to

$$y^2 = 2l\left(\tfrac{1}{2}l - x\right)$$

or, by reflection in the line $x = \frac{1}{4}l$,

8.45 $$y^2 = 2lx.$$

This is the standard equation for the parabola (Figure 8.4c).

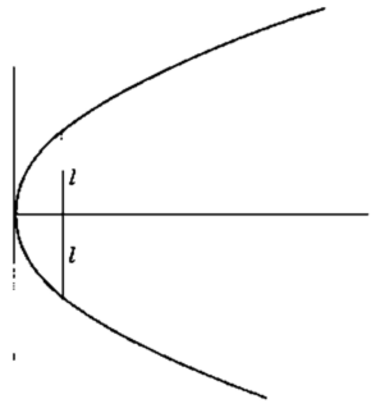


Figure 8.4c

The most convenient parametric equations are: for the ellipse

8.46 $$x = a\cos t, \qquad y = b\sin t,$$

for the parabola

8.47 $$x = 2lt^2, \qquad y = 2lt,$$

and for the hyperbola

8.48 $$x = a\cosh t, \qquad y = b\sinh t,$$

where

$$\cosh t = \frac{e^t + e^{-t}}{2}, \qquad \sinh t = \frac{e^t - e^{-t}}{2}.$$

(These functions will be discussed in § 8.6.)

EXERCISES

1. What kind of curve has the polar equation

$$r = \tfrac{1}{2}l\sec^2\tfrac{1}{2}\theta?$$

2. What kind of curve has the Cartesian equation

$$4x^2 + 24xy + 11y^2 = 5?$$

(See Ex. 5 at the end of §8.2.)

3. The sum (or difference) of the distances of a point on an ellipse (or hyperbola) from the two foci is constant.

4. Express the eccentricity of a central conic in terms of its semiaxes a and b. What is the eccentricity of a rectangular hyperbola?

5. Given points B and C, the locus of the vertex A of a triangle ABC whose Euler line is parallel to BC (as in Ex. 9 at the end of §1.6) is an ellipse whose minor axis is BC while its major axis is twice the altitude of the equilateral triangle on BC. (*Hint*: If A, B, C are (x, y) and $(\mp 1, 0)$, the circumcenter, equidistant from A and C, is $(0, \frac{1}{3}y)$.)

6. An expression such as

$$F = ax^2 + 2hxy + by^2$$

is called a *binary quadratic form*. It is said to be *definite* if $ab > h^2$, so that F has the same sign for all values of x and y except $x = y = 0$. It is said to be *positive* definite if this sign is positive. It is said to be *semidefinite* if $ab = h^2$, so that F is a times a perfect square; *positive* semidefinite if $a > 0$, so that F itself is a perfect square; *indefinite* if $ab < h^2$, so that F is positive for some values of x and y, negative for others. The equation $F = 1$ represents an ellipse if F is positive definite, a pair of parallel lines of F is positive semidefinite, and a hyperbola if F is indefinite.

7. What happens to the equation for the rectangular hyperbola $x^2 - y^2 = a^2$ when we rotate the axes through the angle $\frac{1}{4}\pi$?

8. Describe a geometrical interpretation for the parameter t in 8.46. [*Hint*: Compare $(a \cos t, b \sin t)$ with $(a \cos t, a \sin t)$.]

9. In what respect is the hyperbola 8.441 more satisfactorily represented by the equations

$$x = a \sec u, \qquad y = b \tan u$$

than by the equations 8.48?

10. The circle $r = l$ inverts the conic 8.42 into the limaçon

$$r = l(1 + \varepsilon \cos \theta).$$

Sketch this curve for various values of ε. When $\varepsilon = 1$ (so that the conic is a parabola), it is a cardioid.

11. The circle $r = a$ inverts the rectangular hyperbola $r^2 = a^2 \sec 2\theta$ into the lemniscate of Bernoulli

$$r^2 = a^2 \cos 2\theta$$

(see Ex. 3 at the end of §8.3).

8.5 TANGENT, ARC LENGTH, AND AREA

I do not know what I may appear to the world; but to myself I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me.

Sir Isaac Newton

(*Brewster's Memoirs of Newton*, vol. 2, Chap. 27)

The curves with which we shall be concerned are "rectifiable," that is, there is a well-defined arc length s between any two points P and Q on such a

$$d\frac{y}{x} = \frac{x\,dy - y\,dx}{x^2},$$

the radii (along which y/x remains constant) make a zero contribution to the integral on the right side of 8.57. Hence, if the arc goes from $t = t_1$ to $t = t_2$, the area of the sector is

8.59 $$\tfrac{1}{2}\int_{t_1}^{t_2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt$$

[cf. Courant **1**, p. 273].

EXERCISES

1. The line $x - (t + t')y + 2ltt' = 0$ is a secant of the parabola 8.47, meeting it in the points whose parameters are t and t'. Making t' tend to t, deduce the equation

$$x - 2ty + 2lt^2 = 0$$

for the tangent at the point whose parameter is t, so that, if (x_1, y_1) lies on the parabola 8.45, the tangent at this point is

$$y_1 y = l(x + x_1).$$

2. The line

$$\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = \cos\beta$$

is a secant of the ellipse 8.46, meeting it in the points $t = \alpha \pm \beta$. Making β tend to 0, deduce the equation

$$\frac{x}{a}\cos t + \frac{y}{b}\sin t = 1$$

for the tangent at the point whose parameter is t [Robson **1**, p. 274]. Obtain analogous results for the hyperbola 8.48. Deduce that, if (x_1, y_1) lies on the central conic 8.44, the tangent at this point is

$$\frac{x_1 x}{a^2} \pm \frac{y_1 y}{b^2} = 1.$$

3. At the point t on the ellipse 8.46, the *normal*, being perpendicular to the tangent, is

$$\frac{ax}{\cos t} - \frac{by}{\sin t} = a^2 - b^2.$$

Differentiating partially with respect to t and then eliminating t, obtain the envelope of normals in the form

$$\left(\frac{ax}{a^2 - b^2}\right)^{\frac{2}{3}} + \left(\frac{by}{a^2 - b^2}\right)^{\frac{2}{3}} = 1$$

[Forder **3**, pp. 36–37; Lamb **2**, p. 350]. *Hint*:

$$\cos^3 t = \frac{ax}{a^2 - b^2}, \quad \sin^3 t = -\frac{by}{a^2 - b^2}.$$

4. Use 8.56 to reconcile 8.51 and 8.54.

8.6 HYPERBOLIC FUNCTIONS

The hyperbolic sine and cosine have a property in reference to the rectangular hyperbola, exactly analogous to that of the sine and cosine with reference to the circle. For this reason the former functions are called hyperbolic functions, just as the latter are called circular functions.

E. W. Hobson (1856-1933)
[Hobson **1**, pp. 329-330]

As a very simple application of the formula 8.59, consider the unit circle $x^2 + y^2 = 1$ or

$$x = \cos t, \qquad y = \sin t$$

(Figure 8.5*c*). Since

$$\frac{dx}{dt} = -\sin t = -y \quad \text{and} \quad \frac{dy}{dt} = \cos t = x,$$

the area of the sector from $t = 0$ to any other value is

$$\tfrac{1}{2}\int_0 \left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) dt = \tfrac{1}{2}\int_0 (x^2 + y^2)dt$$

$$= \tfrac{1}{2}\int_0 dt = \tfrac{1}{2}t,$$

which, of course, we knew already. More interestingly (Figure 8.5*d*), if the curve is the rectangular hyperbola $x^2 - y^2 = 1$ or

$$x = \cosh t, \qquad y = \sinh t,$$

so that

$$\frac{dx}{dt} = \sinh t = y \quad \text{and} \quad \frac{dy}{dt} = \cosh t = x,$$

the area of the sector is again

$$\tfrac{1}{2}\int_0 \left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) dt = \tfrac{1}{2}\int_0 (x^2 - y^2)dt$$

$$= \tfrac{1}{2}\int_0 dt = \tfrac{1}{2}t.$$

Comparing the above results, we see clearly the analogy that relates the circular and hyperbolic functions. In Figures 8.5*c* and *d*, we have a sector AOP of the circle or rectangular hyperbola, respectively. In both cases $OA = 1$ and the parameter t is twice the area of the sector. In the former, $OM = \cos t$ and $PM = \sin t$. In the latter, $OM = \cosh t$ and $PM = \sinh t$.

including the ordinary *cylinder of revolution* (or "right circular cylinder") $x^2 + y^2 = k^2$, and the quadric cones

$$ax^2 + by^2 + cz^2 = 0,$$

including the *cone of revolution* (or "right circular cone") $x^2 + y^2 = cz^2$.

The equation $x^2 + y^2 + z^2 = 0$, which is satisfied only by $(0, 0, 0)$, may be regarded either as a peculiar kind of cone or as a sphere of radius zero. The general sphere, having center (x', y', z') and radius k, is, of course,

$$(x - x')^2 + (y - y')^2 + (z - z')^2 = k^2.$$

We observe that this is an equation of the second degree in which the coefficients of x^2, y^2, z^2 are all equal while there are no terms in yz, zx, xy.

The sphere $x^2 + y^2 + z^2 = k^2$, whose center is the origin, inverts the point (X, Y, Z) into

$$\left(\frac{k^2X}{X^2 + Y^2 + Z^2}, \frac{k^2Y}{X^2 + Y^2 + Z^2}, \frac{k^2Z}{X^2 + Y^2 + Z^2}\right).$$

The plane through this inverse point, perpendicular to the line 8.81, namely,

$$Xx + Yy + Zz = k^2,$$

is called the *polar plane* of (X, Y, Z) with respect to the sphere. If (X, Y, Z) lies in the sphere, the polar plane is simply the tangent plane.

The three-dimensional analogues of the conics are the quadric surfaces or *quadrics*, whose plane sections are conics (or occasionally pairs of lines, which may be regarded as degenerate conics). These surfaces, whose equations are of the second degree, include not only the elliptic and hyperbolic cylinders, the quadric cone, and the sphere, but also the *ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

the *hyperboloid of one sheet*

8.86 $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

[R. J. T. Bell **1**, p. 149, Fig. 41], the *hyperboloid of two sheets*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

[Salmon **2**, p. 80, Fig. 1–4], the *elliptic paraboloid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2z,$$

and the *hyperbolic paraboloid*

tain a continuous system of such circles, and a second system by reversing the sign of z.

The plane 8.89 meets the torus in two circles, one in each system (with α replaced by $\alpha + \pi$ in the second system). Since these two circles are sections of the two spheres

$$x^2 + y^2 + z^2 - a^2 + b^2 = \pm 2b(y \cos \alpha - x \sin \alpha),$$

their points of intersection are the two "antipodal" points

$$\left(\pm \frac{a^2 - b^2}{a} \cos \alpha, \ \pm \frac{a^2 - b^2}{a} \sin \alpha, \ \pm \frac{b}{a} \sqrt{a^2 - b^2}\right)$$

(with signs agreeing). Since each of these is a point of contact, 8.89 is a *bitangent* plane [R. J. T. Bell **1**, p. 267].

Comparing 8.89 with 8.87, we see that the "oblique" circles on the torus lie in the same planes (through the center) as the pairs of parallel generators of the hyperboloid of revolution

$$\frac{x^2 + y^2}{a^2 - b^2} - \frac{z^2}{b^2} = 1.$$

(This remark is due to A. W. Tucker.)

EXERCISES

1. The plane through three given points (x_i, y_i, z_i) $(i = 1, 2, 3)$ is

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x & y & z & 1 \end{vmatrix} = 0.$$

If the requirement of passing through a point is replaced (in one or two cases) by the requirement of being parallel to a line with direction numbers X_i, Y_i, Z_i, the corresponding row of the determinant is replaced by

$$X_i \quad Y_i \quad Z_i \quad 0.$$

2. In terms of general Cartesian coordinates, the point (x, y, z) will be transformed into

$(-x, -y, -z)$ by the central inversion $O(-1)$,
$(\mu x, \mu y, \mu z)$ by the dilatation $O(\mu)$ (§7.6),
$(x, y, z+c)$ by a translation along the z-axis.

3. In terms of rectangular coordinates, the point (x, y, z) will be transformed into

$(x, y, -z)$ by reflection in the (x, y)-plane,
(y, x, z) by reflection in the plane $x = y$,
$(-y, x, z)$ by a quarter-turn about the z-axis,
$(x, -y, z+c)$ by a glide reflection (§7.4).

4. In terms of cylindrical coordinates, the point (r, θ, z) will be transformed into

$(r, \theta + \alpha, z + c)$ by a twist,
$(\mu r, \theta + \alpha, \mu z)$ by the general dilative rotation (§7.6).

of R. Cotes* (1682–1716), after whose untimely death Newton said, "If Cotes had lived, we might have known something!"

Setting $\theta = \pi$ in 9.51, we obtain the "famous formula"

$$e^{\pi i} = -1,$$

which connects in such a surprising way the three important numbers

$$e = 2.71828\ldots, \qquad \pi = 3.14159\ldots,$$

and i.

EXERCISES

1. Evaluate $e^{\frac{1}{2}\pi i}$. Is i^i real?

2. From 9.51 deduce $\cos\theta = \dfrac{e^{\theta i} + e^{-\theta i}}{2}$, $\sin\theta = \dfrac{e^{\theta i} - e^{-\theta i}}{2i}$, the familiar formulas for $\cos(\theta + \alpha)$, $\sin(\theta + \alpha)$, and the derivatives of $\cos\theta$, $\sin\theta$.

9.6 ROOTS OF EQUATIONS

Gauss . . . was the first mathematician to use complex numbers in a really confident and scientific way.

G. H. Hardy

[Hardy and Wright **1**, p. 188]

In the field of complex numbers we can solve any quadratic equation that has real coefficients; for example, the equation 9.31 has the two roots i and $-i$. Still more remarkably, we can solve any quadratic equation with complex coefficients; indeed, not only any quadratic equation but also any equation of degree 3 or 4. In saying that we can "solve" an equation we mean here that we can find explicit expressions for the roots in terms of the coefficients. It was proved by E. Galois (who was murdered in 1832 when he was only 20 years old) that the general algebraic equation 9.21 with $n > 4$ cannot be solved in this sense [Infeld **1**]. Nevertheless, the fundamental theorem of algebra (which Gauss proved in 1799) asserts the *existence* of roots for all values of n, even when explicit expressions are not available. (For a neat proof, see Birkhoff and MacLane [**1**, pp. 101–103].) In fact, *numerical* solutions can be found, correct to any assigned number of decimal places.

EXERCISE

A ladder, 24 feet long, rests against a wall with the extra support of a cubical box of edge 7 feet, placed at the bottom of the wall with one horizontal edge against the ladder. How far up the wall does the ladder reach? (*Hint:* Take $7x$ to be the height of the top of the ladder above the top of the box. Obtain an equation whose relevant root is $7x = 9 + 4\sqrt{2}$.)

* *Harmonia mensurarum*, Cambridge, 1722, p. 28.

10
The five Platonic solids

We saw, in §4.6, that the Euclidean plane can be filled with squares, four at each vertex. If we try to fit squares together with only three at each vertex, we find that the figure closes as soon as we have used six squares, and we have a cube $\{4, 3\}$. Similarly, we can fill the plane with equilateral triangles, six at each vertex, and it is interesting to see what happens if we use three, four, or five instead of six. Another possibility is to use pentagons, three at each vertex, in accordance with the symbol $\{5, 3\}$.

With the possible exception of spheres, such *polyhedra* are the simplest solid figures. They provide an easy approach to the subject of topology as well as an interesting exercise in trigonometry. They can be defined and generalized in various ways [see, e.g., Hilbert and Cohn-Vossen **1**, p. 290].

10.1 PYRAMIDS, PRISMS, AND ANTIPRISMS

Although a Discourse of Solid Bodies be an uncommon and neglected Part of Geometry, yet that it is no inconsiderable or unprofitable Improvement of the Science will (no doubt) be readily granted by such, whose Genius tends as well to the Practical as Speculative Parts of it, for whom this is chiefly intended.

Abraham Sharp (1651-1742)

(*Geometry Improv'd*, London, 1717, p. 65)

A *convex polygon* (such as $\{n\}$, where n is an integer) may be described as a finite region of a plane "enclosed" by a finite number of lines, in the sense that its interior lies entirely on one side of each line. Analogously, a *convex polyhedron* is a finite region of space enclosed by a finite number of planes [Coxeter **1**, p. 4]. The part of each plane that is cut off by other planes is a polygon that we call a *face*. Any common side of two faces is an *edge*.

The most familiar polyhedra are *pyramids* and *prisms*. We shall be concerned solely with "right regular" pyramids whose faces consist of a regular n-gon and n isosceles triangles, and with "right regular" prisms whose

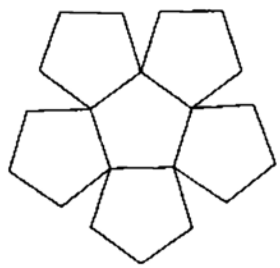


Figure 10.1a

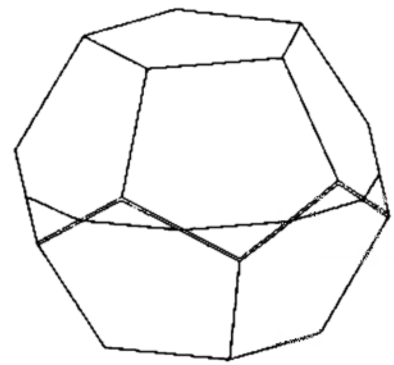


Figure 10.1b

in Table II on p. 413. Each polyhedron is characterized by a Schläfli symbol $\{p, q\}$, which means that it has p-gonal faces, q at each vertex. The numbers of vertices, edges, and faces are denoted by V, E, and F. They can easily be counted in each case, but their significance will become clearer when we have expressed them as functions of p and q. We shall also obtain an expression for the *dihedral angle,* which is the angle between the planes of two adjacent faces.

EXERCISES

1. Give an alternative description of the octahedron (as a dipyramid).

2. Describe a solid having five vertices and six triangular faces.

3. Describe the following sections: (i) of a regular tetrahedron by the plane midway between two opposite edges, (ii) of a cube by the plane midway between two opposite vertices, (iii) of a dodecahedron by the plane midway between two opposite faces.

4. Six congruent rhombi, with angles 60° and 120°, will fit together to form a *rhombohedron* ("distorted cube"). From the two opposite "acute" corners of this solid, regular tetrahedra can be cut off in such a way that what remains is an octahedron. In other words, two tetrahedra and an octahedron can be fitted together to form a rhombohedron. Deduce that the tetrahedron and the octahedron have supplementary dihedral angles, and that infinitely many specimens of these two solids can be fitted together to fill the whole Euclidean space [Ball **1**, p. 147].

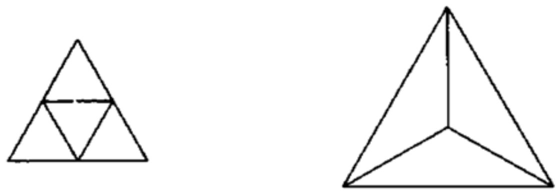
10.2 DRAWINGS AND MODELS

You boil it in sawdust: you salt it in glue:
You condense it with locusts and tape:
Still keeping one principal object in view—
To preserve its symmetrical shape.

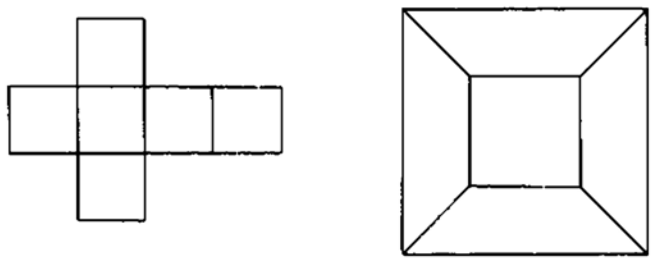
Lewis Carroll

[Dodgson **2a**, Fit 5]

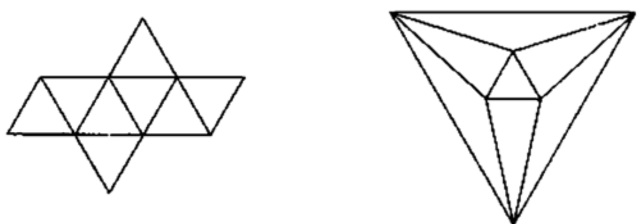
Leonardo da Vinci made skeletal models of polyhedra, using strips of wood for their edges and leaving the faces to be imagined [Pacioli **1**]. When



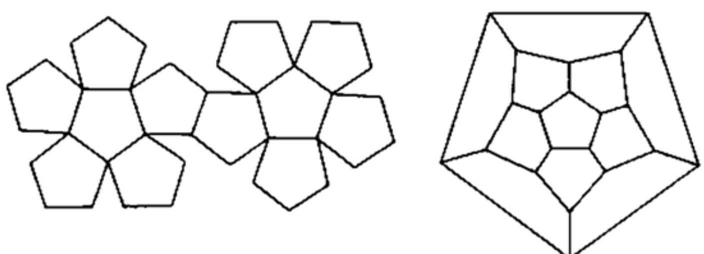
Tetrahedron $\{3, 3\}$



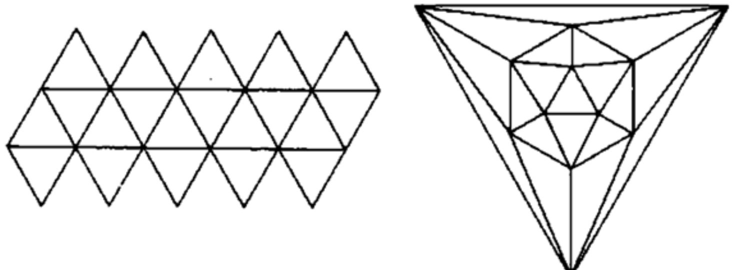
Cube $\{4, 3\}$



Octahedron $\{3, 4\}$



Dodecahedron $\{5, 3\}$



Icosahedron $\{3, 5\}$

Figure 10.2a

$$1 \cdot 1 \quad \text{or} \quad 2 \cdot 1 \quad \text{or} \quad 1 \cdot 2 \quad \text{or} \quad 3 \cdot 1 \quad \text{or} \quad 1 \cdot 3.$$

These five possibilities provide a simple proof of Euclid's assertion [Rademacher and Toeplitz **1**, pp. 84–87]:

There are just five convex regular polyhedra:

$$\{3, 3\}, \quad \{4, 3\}, \quad \{3, 4\}, \quad \{5, 3\}, \quad \{3, 5\}.$$

The inequality 10.33 is not merely a necessary condition for the existence of $\{p, q\}$ but also a sufficient condition; for in § 10.1 we saw how to construct a solid corresponding to each solution.

The same inequality arises in a more elementary manner when we construct a model of the polyhedron from its net. At a vertex we have q p-gons, each contributing an angle

$$\left(1 - \frac{2}{p}\right)\pi.$$

In order to form a solid angle, these q face angles must make a total less than 2π. Thus

$$q\left(1 - \frac{2}{p}\right)\pi < 2\pi$$

whence, as before, $(p - 2)(q - 2) < 4$.

Any maker of models soon observes that the amount by which the sum of the face angles at a vertex falls short of 2π is smaller for a complicated solid like the dodecahedron than for a simple one like the tetrahedron. Descartes proved that if this amount, say δ, is the same at every vertex, it is actually equal to $4\pi/V$ [Brückner **1**, p. 60]. In the case of $\{p, q\}$, this is an immediate consequence of the formula 10.32 for V, which yields

$$\frac{4\pi}{V} = (2p + 2q - pq)\frac{\pi}{p} = 2\pi - q\left(1 - \frac{2}{p}\right)\pi.$$

EXERCISES

1. The number of edges of $\{p, q\}$ is given by

$$E^{-1} = p^{-1} + q^{-1} - \tfrac{1}{2}.$$

2. Consider an arbitrary polyhedron having p-gonal faces for various values of p, and q faces at a vertex for various values of q. Generalize the equations 10.31 in the form

$$\Sigma q = 2E = \Sigma p,$$

where the first summation is taken over all the vertices and the last over all the faces. Deduce that every polyhedron has either a face with $p = 3$ or a vertex with $q = 3$ (or both). (*Hint*: If not, we would have $\Sigma q \geqslant 4V$ and $\Sigma p \geqslant 4F$.)

3. If the faces are all alike and the edges are all alike and the vertices are all alike, the faces are regular. Show by an example that this result for polyhedra is not valid for tessellations.

6. The points (x, y, z) that belong to the solid octahedron are given by the inequality

$$|x| + |y| + |z| \leqslant 1.$$

7. If each edge of a regular tetrahedron is projected into an arc of a great circle on the circumsphere, at what angles do these arcs intersect? Find the equations of the planes of the six great circles.

The figure of a pentagon with diagonals can be neatly displayed by tying a simple knot in a long strip of paper and carefully pressing it flat (Figure 11.1*c*).

EXERCISES

1. Defining τ to be the ratio of the diagonal of a regular pentagon to its side, establish 11.11 by applying 1.54 to the isosceles triangle PRS.

2. Show how one further setting of the compasses will yield a point (in Figure 11.1*b*) dividing the given segment QU in the ratio $\tau : 1$.

11.2 DE DIVINA PROPORTIONE

Del suo secondo essentiale effecto . . . Del terço suo singulare effecto . . . Del quarto suo ineffabile effecto . . . . . . Del .10. suo supremo effecto . . . Del suo .11. excellentissimo effecto . . . Del suo .12. quasi incomprehensibile effecto . . .

Luca Pacioli (ca. 1445-1509)
[Pacioli **1**, pp. 6-7]

Under the beneficent influence of the artist Piero della Francesca (ca. 1416–1492), Fra Luca Pacioli (or *Paccioli*) wrote a book about τ, called *De divina proportione,* which he illustrated with drawings of models made by his friend Leonardo da Vinci. His enthusiasm for the subject is apparent in the above titles that he chose for various chapters. (It is interesting to observe how closely some of his old Italian resembles English.)

"The seventh inestimable effect" is the occurrence of τ as the circumradius of a regular decagon of side 1. (We can thus inscribe a pentagon in a given circle by first inscribing a decagon and then picking out alternate vertices.) "The ninth effect, the best of all" is that two crossing diagonals of a regular pentagon divide one another in extreme and mean ratio. "The twelfth almost incomprehensible effect" is the following property of the regular icosahedron $\{3, 5\}$.

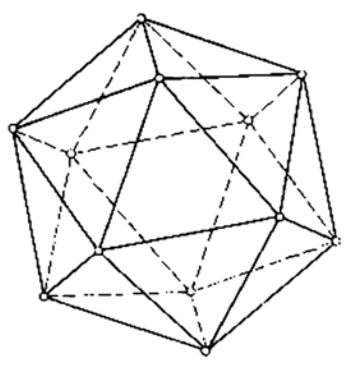


Figure 11.2a

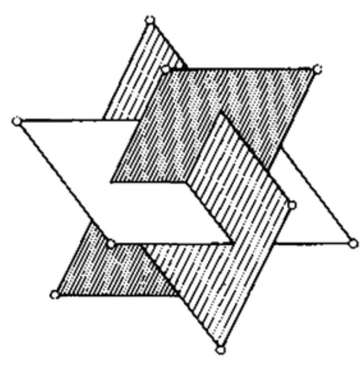


Figure 11.2b

The faces surrounding a vertex of the icosahedron belong to a pyramid whose base is a regular pentagon (similar to the vertex figure). Any two opposite edges of the icosahedron belong to a rectangle whose longer sides are diagonals of such pentagons. Since the diagonal of a pentagon is τ times its side, this rectangle is a *golden* rectangle, whose sides are in the ratio $\tau : 1$. In fact, the twelve vertices of the icosahedron (Figure 11.2*a*) are the twelve vertices of three golden rectangles in mutually perpendicular planes (Figure 11.2*b*). A model is easily made from three ordinary postcards (which are nearly golden rectangles). In the middle of each card, cut a slit parallel to the long sides and equal in length to the short sides. For a practical reason, the slit in one of the cards must be continued right to the edge. Then the cards can easily be put together so that each passes through the middle of another, in cyclic order.

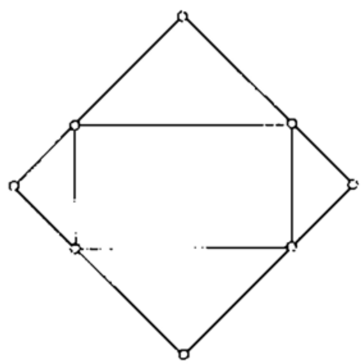


Figure 11.2c

We see from Figure 11.2*c* that a golden rectangle can be inscribed in a square so that each vertex of the rectangle divides a side of the square in the ratio $\tau : 1$. Identifying this with one of the three "equatorial" squares of a regular octahedron, we deduce that an icosahedron can be inscribed in an octahedron so that each vertex of the icosahedron divides an edge of the octahedron in the ratio $\tau : 1$.

EXERCISES

1. Using Cartesian coordinates referred to the planes of the three golden rectangles, obtain the 12 vertices of the icosahedron in the form

$$(0, \pm\tau, \pm 1), \quad (\pm 1, 0, \pm\tau), \quad (\pm\tau, \pm 1, 0).$$

2. These 12 points divide the 12 edges of the octahedron

$$(\pm\tau^2, 0, 0), \quad (0, \pm\tau^2, 0), \quad (0, 0, \pm\tau^2)$$

in the ratio $\tau : 1$.

3. Obtain coordinates for the 20 vertices of the regular dodecahedron [Coxeter **1**, p. 53].

$$f_{2k-1} = f_{k-1}^2 + f_k^2 \quad \text{and} \quad f_{2k} = f_k g_k,$$

obtaining
$$\begin{aligned} f_{2k+1} &= f_{k-1}^2 + f_k(f_{k-1} + f_{k+1}) + f_k^2 \\ &= (f_{k-1} + f_k)f_{k+1} + f_k^2 \\ &= f_k^2 + f_{k+1}^2, \end{aligned}$$

and then add

$$f_{2k} = f_k g_k \quad \text{and} \quad f_{2k+1} = f_k^2 + f_{k+1}^2,$$

obtaining
$$\begin{aligned} f_{2k+2} &= f_k^2 + f_k(f_{k-1} + f_{k+1}) + f_{k+1}^2 \\ &= f_k f_{k+1} + f_{k+1} f_{k+2} \\ &= f_{k+1} g_{k+1}. \end{aligned}$$

Similarly, to establish the identity

11.45 $$\tau^n = f_n\tau + f_{n-1},$$

which is obvious when $n = 1$ or 2, we add

$$\tau^k = f_k\tau + f_{k-1} \quad \text{and} \quad \tau^{k+1} = f_{k+1}\tau + f_k,$$

obtaining
$$\tau^{k+2} = f_{k+2}\tau + f_{k+1}.$$

The identity 11.45 continues to hold when n is negative, provided we define $f_{-k} = f_{-k+2} - f_{-k+1}$ (for $k > 0$), so that

$$f_{-k} = (-1)^{k+1} f_k.$$

Thus
$$\tau^{-k} = f_{-k}\tau + f_{-k-1}$$

11.46 $$= (-1)^{k+1}(f_k\tau - f_{k+1}).$$

and

11.47 $$(-\tau)^{-k} = f_{k+1} - f_k\tau$$

11.471 $$= f_{k-1} - f_k\tau^{-1}.$$

Adding 11.45 (with k for n) and 11.47, we obtain

11.48 $$g_k = \tau^k + (-\tau)^{-k}.$$

Similarly, subtracting 11.471 from 11.45 (with k for n), we obtain

11.49 $$f_k = \frac{\tau^k - (-\tau)^{-k}}{\tau + \tau^{-1}} = \frac{\tau^k - (-\tau)^{-k}}{\sqrt{5}},$$

an explicit formula which was discovered by J. P. M. Binet in 1843.

From 11.48 and 11.49 we immediately deduce

$$\tau^k = \frac{g_k + f_k\sqrt{5}}{2}, \quad (-\tau)^{-k} = \frac{g_k - f_k\sqrt{5}}{2};$$

Part III

two rays and $n-1$ segments. The points can be named $P_1, P_2, \ldots, P_n$ in such a way that the two rays are P_1/P_n, P_n/P_1, and the $n-1$ segments are

$$P_1P_2,\ P_2P_3,\ \ldots,\ P_{n-1}P_n,$$

each containing none of the points. We say that the points are in the *order* $P_1P_2 \ldots P_n$, and write $[P_1P_2 \ldots P_n]$. Necessary and sufficient conditions for this are

$$[P_1P_2P_3],\quad [P_2P_3P_4],\quad \ldots,\quad [P_{n-2}P_{n-1}P_n].$$

Naturally, the best logical development of any subject uses the simplest or "weakest" possible set of axioms. (The worst occurs when we go to the opposite extreme and assume everything, so that there is no development at all!) In his original formulation of Axiom 12.27 [Pasch and Dehn **1**, p. 2: "IV. Kernsatz"] Pasch made the following far stronger statement: If a line in the plane of a given triangle meets one side, it also meets another side (or else passes through a vertex). Peano's formulation, which we have adopted, excels this in two respects. The word "plane" (which we shall define in § 12.4) is not used at all, and the line DE penetrates the triangle ABC in a special manner, namely, before entering through the side CA, it comes from a point D on C/B. It might just as easily have come from a point on A/B (which is the same with C and A interchanged) or from a point on B/A or B/C (which is quite a different story). The latter possibility (with a slight change of notation) is covered by the following theorem (12.278). Axiom 12.27 is "only just strong enough"; for, although it enables us to deduce the statement 12.278 of apparently equal strength, we could not reverse the roles: if we tried instead to use 12.278 as an axiom, we would not be able to deduce 12.27 as a theorem!

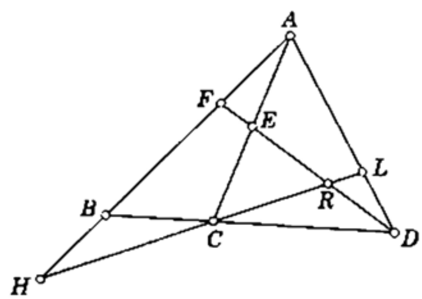


Figure 12.2d

THEOREM 12.278 *If ABC is a triangle and $[AFB]$ and $[BCD]$, then there is, on the line DF, a point E for which $[CEA]$.*

Proof. Take H on B/F (as in Figure 12.2*d*) and consider the triangle DFB with $[FBH]$ and $[BCD]$. By 12.27 and 12.272, there is a point R for which $[DRF]$ and $[HCR]$. By 12.274, $[AFB]$ and $[FBH]$ imply $[AFH]$. Thus we have a triangle DAF with $[AFH]$ and $[FRD]$. By 12.27 and 12.272 again, there is a point L for which $[DLA]$ and $[HRL]$. By 12.277, $[HCR]$ and $[HRL]$

We naturally regard this ratio as being negative if P lies between Q and Q', that is, if the two triangles are oppositely oriented.

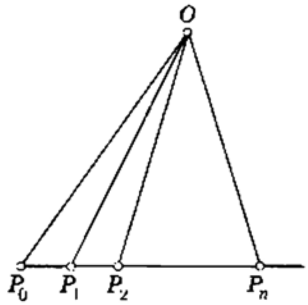


Figure 13.4c

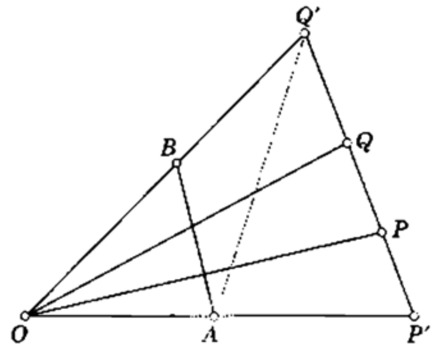


Figure 13.4d

These ideas enable us to define the area of any polygon in such a way that *equivalent polygons have the same area*, and when two polygons are stuck together to make a larger polygon, the areas are added. To compute the area of a given polygon in terms of a standard triangle OAB as unit of measurement, we dissect the polygon into triangles and add the areas of the pieces, each computed as follows.

By applying a suitable translation, any given triangle can be shifted so that one vertex coincides with the vertex O of the standard triangle OAB. Accordingly, we consider a triangle OPQ. Let the line PQ meet OA in P', and OB in Q', as in Figure 13.4*d*. Multiplying together the three ratios

$$\frac{OPQ}{OP'Q'} = \frac{PQ}{P'Q'}, \quad \frac{OP'Q'}{OAQ'} = \frac{OP'}{OA}, \quad \frac{OAQ'}{OAB} = \frac{OQ'}{OB}$$

we obtain the desired ratio

13.43
$$\frac{OPQ}{OAB} = \frac{PQ}{P'Q'}\frac{OP'}{OA}\frac{OQ'}{OB}.$$

To obtain an analytic expression for the area of a triangle OPQ, referred to axes through the vertex O, we take the coordinates of the points

$$O, \quad A, \quad B, \quad P, \quad Q, \quad P', \quad Q'$$

to be

$$(0, 0), (1, 0), (0, 1), (x_1, y_1), (x_2, y_2), (p, 0), (0, q),$$

respectively. Since the equation

$$\frac{x}{p} + \frac{y}{q} = 1$$

and can serve as a unit cell. If the vertices of such a parallelogram (in counterclockwise order) are

$$(0, 0), \quad (x, y), \quad (x + x_1, y + y_1), \quad (x_1, y_1),$$

we see from 13.44 that

13.52 $$xy_1 - yx_1 = 1.$$

In other words, this is the condition for the points

13.53 $$(0, 0), \quad (x, y), \quad (x_1, y_1)$$

to form a positively oriented "empty" triangle of area $\frac{1}{2}$, which could be used just as well as (0, 0) (1, 0) (0, 1) to generate the lattice. Thus a lattice is completely determined, apart from its position, by the area of its unit cell. Moreover, although there are infinitely many visible points in a given lattice, they all play the same role. (These properties of affine geometry are in marked contrast to Euclidean geometry, where the shape of a lattice admits unlimited variation and each lattice contains visible points at infinitely many different distances.)

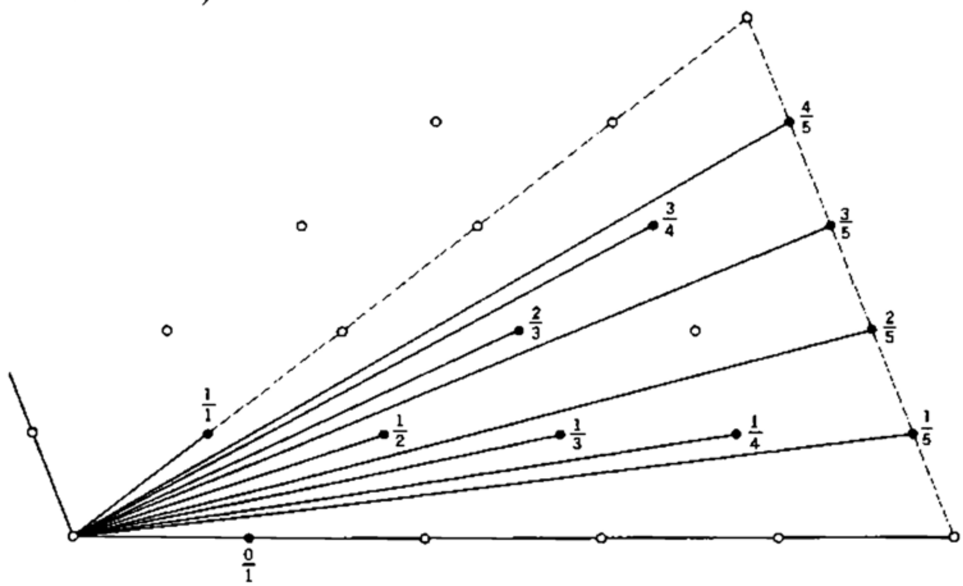


Figure 13.5b

George Pólya* has applied 13.52 to a useful lemma in the theory of numbers. The *Farey series* F_n of order n is the ascending sequence of fractions from 0 to 1 whose denominators do not exceed n. Thus y/x belongs to F_n if x and y are coprime and

13.54 $$0 \leqslant y \leqslant x \leqslant n.$$

For instance, F_5 is

$$\tfrac{0}{1}, \tfrac{1}{5}, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{2}{5}, \tfrac{1}{2}, \tfrac{3}{5}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{4}{5}, \tfrac{1}{1}.$$

* *Acta Litterarum ac Scientiarum Regiae Universitatis Hungaricae Francisco-Josephinae*, Sectio

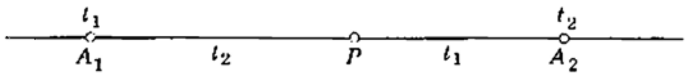


Figure 13.7a

Conversely, given a point P on the line A_1A_2, we can find numbers t_1 and t_2 such that

$$\frac{t_2}{t_1} = \frac{A_1P}{PA_2} \quad \text{or} \quad \frac{t_1}{t_2} = \frac{PA_2}{A_1P};$$

then P will be the centroid of masses t_1 and t_2 at A_1 and A_2. Since masses μt_1 and μt_2 (where $\mu \neq 0$) determine the same point as t_1 and t_2, these *barycentric coordinates* are homogeneous:

$$(t_1, t_2) = (\mu t_1, \mu t_2) \qquad (\mu \neq 0).$$

Similarly, as Möbius observed in 1827, we may set up barycentric coordinates in the plane of a *triangle of reference* $A_1A_2A_3$. If $t_1 + t_2 + t_3 \neq 0$, masses t_1, t_2, t_3 at the three vertices determine a point P (the centroid) whose coordinates are (t_1, t_2, t_3). In particular, $(1, 0, 0)$ is A_1, $(0, 1, 0)$ is A_2, $(0, 0, 1)$ is A_3, and $(0, t_2, t_3)$ is the point on A_2A_3 whose one-dimensional coordinates with respect to A_2 and A_3 are (t_2, t_3). To find coordinates for a given point P of general position, we find t_2 and t_3 from such a point Q on the line A_1P, as in Figure 13.7*b*, and then determine t_1 as the mass at A_1 that will balance a mass $t_2 + t_3$ at Q so as to make P the centroid. Again, as in the one-dimensional case, these coordinates are homogeneous:

$$(t_1, t_2, t_3) = (\mu t_1, \mu t_2, \mu t_3) \qquad (\mu \neq 0).$$

Joining P to A_1, A_2, A_3, we decompose $A_1A_2A_3$ into three triangles having a common vertex P. *The areas of these triangles are proportional to the barycentric coordinates of* P, as in Figure 13.7*c*. This fact follows at once from 13.42, since

$$\frac{t_3}{t_2} = \frac{A_2Q}{QA_3} = \frac{A_1A_2Q}{A_1QA_3} = \frac{PA_2Q}{PQA_3} = \frac{A_1A_2Q - PA_2Q}{A_1QA_3 - PQA_3} = \frac{PA_1A_2}{PA_3A_1}.$$

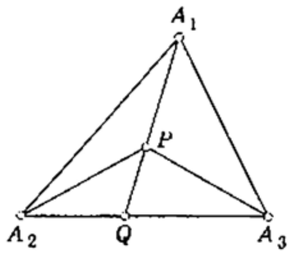


Figure 13.7b

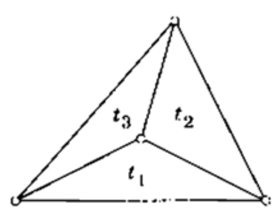


Figure 13.7c

Proof [Forder **1**, p. 140]. When all three lines are in one plane, this follows at once from 13.11, so let us assume that they are not. For any point Q on q, the planes Qp and Qr meet in a line, say q' (Figure 13.8a). Any common point of q' and r would lie in both the planes Qp, pr, and therefore on their common line p; this is impossible, since p is parallel to r. Hence q' is parallel to r. But the only line through Q parallel to r is q. Hence q coincides with q', and is coplanar with p. Any common point of p and q would lie also on r. Hence p and q are parallel.

The transitivity of parallelism provides an alternative proof for 13.81. To establish the impossibility of a point O lying on both planes γ and $q'r'$, we imagine two lines through O, parallel to q (and q'), r (and r'). The planes γ and $q'r'$, each containing both these lines, would coincide, contradicting our assumption that A does not lie in γ.

The three face planes OBC, OCA, OAB of a tetrahedron $OABC$ form with the respectively parallel planes through A, B, C a *parallelepiped* whose faces are six parallelograms, as in Figure 13.8b [Forder **1**, p. 155].

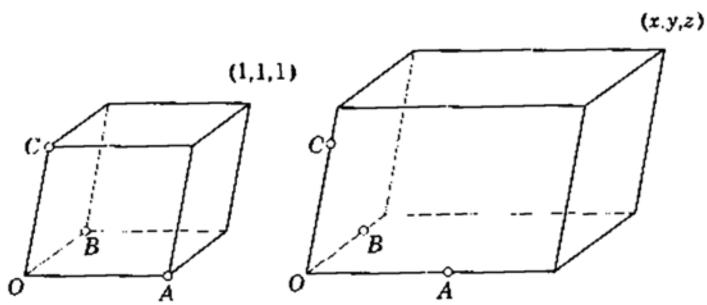


Figure 13.8b **Figure 13.8c**

It is now easy to build up a three-dimensional theory of dilatations, translations, and vectors. Three vectors $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$ are said to be *dependent* if they are coplanar, in which case each is expressible as a linear combination of the other two. Three vectors $\mathbf{e}$, $\mathbf{f}$, $\mathbf{g}$ are said to be *independent* if the only solution of the vector equation

$$x\mathbf{e} + y\mathbf{f} + z\mathbf{g} = \mathbf{0}$$

is $x = y = z = 0$. Three such vectors provide a basis for a system of three-dimensional *affine coordinates*. In fact, if

$$\mathbf{e} = \overrightarrow{OA},\ \mathbf{f} = \overrightarrow{OB},\ \mathbf{g} = \overrightarrow{OC},$$

as in Figure 13.8c, the general vector $\overrightarrow{OP}$ may be exhibited as a diagonal of the parallelepiped formed by drawing through P three planes parallel to OBC, OCA, OAB. Then

$$\overrightarrow{OP} = x\mathbf{e} + y\mathbf{f} + z\mathbf{g},$$

where the terms of this sum are vectors along three edges of the parallelepiped.

In space, as in a plane, the centroid P of masses t_i at points A_i is determined by a vector $\overrightarrow{OP}$ such that

$$\Sigma t_i\, \overrightarrow{OP} = \Sigma t_i\, \overrightarrow{OA_i} \qquad (\Sigma t_i \neq 0).$$

If $\overrightarrow{OA_i} = x_i\mathbf{e} + y_i\mathbf{f} + z_i\mathbf{g}$, we deduce

$$\Sigma t_i\, \overrightarrow{OP} = \Sigma t_i x_i \mathbf{e} + \Sigma t_i y_i \mathbf{f} + \Sigma t_i z_i \mathbf{g}.$$

Hence, in terms of affine coordinates,

13.83 *The centroid of k masses t_i $(\Sigma t_i \neq 0)$ at points (x_i, y_i, z_i) $(i = 1, \ldots, k)$ is*

$$\left(\frac{\Sigma t_i x_i}{\Sigma t_i}, \quad \frac{\Sigma t_i y_i}{\Sigma t_i}, \quad \frac{\Sigma t_i z_i}{\Sigma t_i}\right).$$

In particular, if $t_1 + t_2 + t_3 = 1$, the centroid of three masses t_1, t_2, t_3 at the points

$$(1, 0, 0), \quad (0, 1, 0), \quad (0, 0, 1)$$

is (t_1, t_2, t_3). Hence

13.84 *The affine coordinates of any point in the plane $x + y + z = 1$ are the same as its areal coordinates referred to the triangle cut out from this plane by the coordinate planes $x = 0,\ y = 0,\ z = 0$.*

It follows that there is a line

$$\frac{x}{t_1} = \frac{y}{t_2} = \frac{z}{t_3}$$

through the origin (in affine space) for each point with barycentric coordinates (t_1, t_2, t_3). On the other hand, lines lying in the plane $x + y + z = 0$ yield no corresponding points in the parallel plane $x + y + z = 1$, unless we agree to extend the affine plane by postulating a line at infinity

$$t_1 + t_2 + t_3 = 0$$

so as to form the projective plane. This possibility has already been mentioned in § 6.9; we shall explore it more systematically in Chapter 14.

EXERCISES

1. If a line a is parallel to a plane α, and a plane through a meets α in b, then a and b are parallel lines. If another plane through a meets α in c, then b and c are parallel lines.

2. If α, β, γ are planes intersecting in lines $\beta \cdot \gamma = a,\ \ \gamma \cdot \alpha = b,\ \ \alpha \cdot \beta = c$, and a is parallel to b, then a, b, c are all parallel.

3. All the lines through A parallel to α are in a plane parallel to α [Forder **1**, p. 155].

4. Each of the six edges of a tetrahedron lies on a plane joining this edge to the midpoint of the opposite edge. The six planes so constructed all pass through one point: the centroid of equal masses at the four vertices.

5. Develop the theory of three-dimensional barycentric coordinates referred to a tetrahedron $A_1A_2A_3A_4$.

13.9 THREE-DIMENSIONAL LATTICES

The small parallelepiped built upon the three translations selected as unit translations . . . is known as the unit cell. . . . The entire crystal structure is generated through the periodic repetition, by the three unit translations, of the matter contained within the volume of the unit cell.

M. J. Buerger (1903 -)

[Buerger **1**, p. 5]

The theory of volume in affine space is more difficult than that of area in the affine plane, because of the complication introduced by M. Dehn's observation that two polyhedra of equal volume are not necessarily derivable from each other by dissection and rearrangement. A valid treatment, suggested by Mrs. Sally Ruth Struik, may be described very briefly as follows. It is found that any two tetrahedra are related by a unique *affinity* $ABCD \to A'B'C'D'$, which transforms the whole space into itself in such a way as to preserve collinearity. In particular, a tetrahedron $ABCC'$ is transformed into $ABC'C$ by the *affine reflection*

$$AB(CC'),$$

which interchanges C and C' while leaving invariant every point in the plane that joins AB to the midpoint of CC'. Two tetrahedra are said to have the same *volume* if one can be transformed into the other by an *equiaffinity*: the product of an even number of affine reflections. Such a comparison is easily extended from tetrahedra to parallelepipeds, since a parallelepiped can be dissected into six tetrahedra all having the same volume.

In three dimensions, as in two, a *lattice* may be regarded as the set of points whose affine coordinates are integers. However, as it is independent of the chosen coordinate system, it is more symmetrically described as a discrete set of points whose set of position vectors is *closed under subtraction,* that is, along with any two of the vectors the set includes also their difference. Subtracting any one of the vectors from itself, we obtain the zero vector

$$\mathbf{c} - \mathbf{c} = \mathbf{0}$$

and hence also $\mathbf{0} - \mathbf{b} = -\mathbf{b}$, $\mathbf{a} - (-\mathbf{b}) = \mathbf{a} + \mathbf{b}$, $\mathbf{a} + \mathbf{a} = 2\mathbf{a}$, and so on: the set of vectors, containing the difference of any two, also contains the sum of any two, and all the integral multiples of any one. The lattice is one-,

two-, or three-dimensional according to the number of independent vectors. In the three-dimensional case, a set of three independent vectors $\mathbf{e}, \mathbf{f}, \mathbf{g}$ is called a *basis* for the lattice if all the vectors are expressible in the form

13.91 $$x\mathbf{e} + y\mathbf{f} + z\mathbf{g},$$

where x, y, z are integers. If three of these vectors, say $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ form another basis for the same lattice, there must exist 18 integers

$$a_\alpha, b_\alpha, c_\alpha, A_\alpha, B_\alpha, C_\alpha \qquad (\alpha = 1, 2, 3)$$

such that

$$\mathbf{r}_\alpha = a_\alpha \mathbf{e} + b_\alpha \mathbf{f} + c_\alpha \mathbf{g}, \quad \mathbf{e} = \Sigma A_\alpha \mathbf{r}_\alpha, \quad \mathbf{f} = \Sigma B_\alpha \mathbf{r}_\alpha, \quad \mathbf{g} = \Sigma C_\alpha \mathbf{r}_\alpha$$

and therefore

$$\mathbf{r}_\alpha = a_\alpha \Sigma A_\beta \mathbf{r}_\beta + b_\alpha \Sigma B_\beta \mathbf{r}_\beta + c_\alpha \Sigma C_\beta \mathbf{r}_\beta,$$

whence

$$a_\alpha A_\beta + b_\alpha B_\beta + c_\alpha C_\beta = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta. \end{cases}$$

Since the product of two determinants is obtained by combining the rows of one with the columns of the other, we have

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

Since the two determinants on the left are integers whose product is 1, each must be ± 1. Conversely, if $a_\alpha, b_\alpha, c_\alpha$ are given so that their determinant is ± 1, we can derive $A_\alpha, B_\alpha, C_\alpha$ by "inverting the matrix," and the given basis $\mathbf{e}, \mathbf{f}, \mathbf{g}$ yields the equally effective basis $\mathbf{r}_\alpha$. Hence

A necessary and sufficient condition for two triads of independent vectors

$$\mathbf{e}, \mathbf{f}, \mathbf{g} \qquad \textit{and} \qquad a_\alpha \mathbf{e} + b_\alpha \mathbf{f} + c_\alpha \mathbf{g} \qquad (\alpha = 1, 2, 3)$$

to be alternative bases for the same lattice is

13.92 $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \pm 1$$

[cf. Hardy and Wright **1**, p. 28; Neville **1**, p. 5].

In other words, a lattice is derived from any one of its points by applying a *discrete group of translations:* one-, two-, or three-dimensional according as the translations are collinear, coplanar but not collinear, or not coplanar. In the one-dimensional case the generating translation is unique (except that it may be reversed), but in the other cases the two or three generators, that is, the basic vectors, may be chosen in infinitely many ways. When they have

been chosen, we can use them to set up a system of affine coordinates so that, in the three-dimensional case, the vector 13.91 goes from the origin (0, 0, 0) to the point (x, y, z), and the lattice consists of the points whose coordinates are integers. The eight points

$$(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1), (0, 1, 1), (1, 0, 1), (1, 1, 0), (1, 1, 1),$$

derived from the eight vectors

$$\mathbf{0}, \quad \mathbf{e}, \quad \mathbf{f}, \quad \mathbf{g}, \quad \mathbf{f} + \mathbf{g}, \quad \mathbf{g} + \mathbf{e}, \quad \mathbf{e} - \mathbf{f}, \quad \mathbf{e} + \mathbf{f} + \mathbf{g},$$

evidently form a parallelepiped, which is a *unit cell* of the lattice. By an argument analogous to that used for a two-dimensional lattice in §4.1, *any two unit cells for the same lattice have the same volume.*

Any line joining two of the lattice points contains infinitely many of them, forming a one-dimensional sublattice of the three-dimensional lattice. In fact, the line joining (0, 0, 0) and (x, y, z) contains also (nx, ny, nz) for every integer n. If x, y, z have the greatest common divisor d, the lattice point

$$(x/d, y/d, z/d)$$

lies on this same line, and the corresponding translation generates the group of the one-dimensional lattice. The lattice point (x, y, z) is *visible* if and only if the three integers x, y, z have no common divisor greater than 1.

Any triangle of lattice points determines a plane containing a two-dimensional sublattice. For, if vectors

$$\mathbf{r}_1 = x_1\mathbf{e} + y_1\mathbf{f} + z_1\mathbf{g} \quad \text{and} \quad \mathbf{r}_2 = x_2\mathbf{e} + y_2\mathbf{f} + z_2\mathbf{g}$$

have integral components, so also does $t_1\mathbf{r}_1 + t_2\mathbf{r}_2$ for any integers t_1 and t_2. The parallel plane through any other lattice point will contain a congruent sublattice. Thus we may regard all the lattice points as being distributed among an infinite sequence of parallel planes, called *rational planes* [Buerger **1**, p. 7].

Any such plane, being the join of three points whose coordinates are integers, has an equation of the form

13.93 $$Xx + Yy + Zz = N,$$

where the coefficients X, Y, Z, N are integers, so that the intercepts on the coordinate axes have the rational values $N/X, N/Y, N/Z$. (This is the reason for the name "rational" planes.) We may assume that the greatest common divisor of X, Y, Z is 1; for, any common factor of X, Y, Z would be a factor of N too, and then we could divide both sides of the equation by this number, obtaining a simpler and equally effective equation for the same plane.

Conversely, any such equation (in which the greatest common divisor of X, Y, Z is 1) represents a plane containing a two-dimensional sublattice. This is obvious when $X = 1$, since then we can assign arbitrar integral

values to y, z, and solve 13.93 for x. When X, Y, Z are all greater than 1, we consider the set of numbers

$$xX + yY + zZ,$$

where x, y, z are variable integers while X, Y, Z remain constant. This set (like the set of lattice vectors) is an *ideal:* it contains the difference of any two of its members and (therefore) all the multiples of any one. Let d denote its smallest positive member, and N any other member. Then N is a multiple of d: for otherwise we could divide N by d and obtain a remainder $N - qd$, which would be a member smaller than d. Thus every member of the set is a multiple of d. But X, Y, Z are members. Therefore d, being a common divisor, must be equal to 1, and the set simply consists of all the integers. In other words, the equation 13.93 has one integral solution (and therefore infinitely many) [cf. Uspensky and Heaslet **1**, p. 54].

For each triad of integers X, Y, Z, coprime in the above sense (but not necessarily coprime in pairs), we have a sequence of parallel planes 13.93, evenly spaced, one plane for each integer N. Since every lattice point lies in one of the planes, the infinite region between any two consecutive planes is completely empty. One of the planes, namely that for which $N = 0$, passes through the origin. The nearest others, given by $N = \pm 1$, are appropriately called *first* rational planes [Buerger **1**, p. 9]. We shall have occasion to consider them again in § 18.3.

EXERCISES

1. How can a parallelepiped be dissected into six tetrahedra all having the same volume?

2. Identify the transformation $(x, y, z) \to (x, y, -z)$ with the affine reflection that leaves invariant the plane $z = 0$ while interchanging the points $(0, 0, \pm 1)$.

3. A lattice is transformed into itself by the central inversion that interchanges two of its points.

4. Every lattice point in a first rational plane is visible.

5. Is every rational plane through a visible point a first rational plane?

6. Find a triangle of lattice points in the first rational plane

$$6x + 10y + 15z = 1.$$

7. Obtain a formula for all the lattice points in this plane.

8. The origin is the only lattice point in the plane

$$x + \sqrt{2}y + \sqrt{3}z = 0.$$

14
Projective geometry

In affine geometry, as we have seen, parallelism plays a leading role. In projective geometry, on the other hand, there is no parallelism: every pair of coplanar lines is a pair of intersecting lines. The conflict with 12.61 is explained by the fact that the projective plane is not an "ordered" plane. The set of points on a line, like the set of lines through a point, is closed: given three, we cannot pick out one as lying "between" the other two. At first sight we might expect a geometry having no circles, no distances, no angles, no intermediacy, and no parallelism, to be somewhat meagre. But, in fact, a very beautiful and intricate collection of propositions emerges: propositions of which Euclid never dreamed, because his interest in measurement led him in a different direction. A few of these nonmetrical propositions were discovered by Pappus of Alexandria in the fourth century A.D. Others are associated with the names of two Frenchmen: the architect Girard Desargues (1591–1661) and the philosopher Blaise Pascal (1623–1662). Meanwhile, the related subject of perspective [Yaglom **2,** p. 31] had been studied by artists such as Leonardo da Vinci (1452–1519) and Albrecht Dürer (1471–1528).

Kepler's invention of points at infinity made it possible to regard the projective plane as the affine plane plus the line at infinity. A converse relationship was suggested by Poncelet's *Traité des propriétés projectives des figures* (1822) and von Staudt's *Geometrie der Lage* (1847), in which projective geometry appeared as an independent science, making it possible to regard the affine plane as the projective plane minus an arbitrary line o, and then to regard the Euclidean plane as the affine plane with a special rule for associating pairs of points on o (in "perpendicular directions") [Coxeter **2,** pp. 115, 138]. This standpoint became still clearer in 1899, when Mario Pieri placed the subject on an axiomatic foundation. Other systems of axioms, slightly different from Pieri's, have been proposed by subsequent authors. The particular system that we shall give in § 14.1 was suggested by Bachmann [**1,** pp. 76–77]. To test the consistency of a system of axioms, we apply it to a "model," in which the primitive concepts are represented by familiar concepts whose properties we are prepared to accept [Coxeter **2,** pp.

186–187]. In the present case a convenient model for the projective plane is provided by the affine plane plus the line at infinity (§ 6.9). We shall extend the barycentric coordinates of § 13.7 to general projective coordinates, so as to eliminate the special role of the line at infinity. The result may be regarded as a purely algebraic model in which a *point* is an ordered triad of numbers (x_1, x_2, x_3), not all zero, with the rule that $(\mu x_1, \mu x_2, \mu x_3)$ is the same point for any $\mu \neq 0$, and a *line* is a homogeneous linear equation. One advantage of this model is that the numbers x_α and μ are not necessarily real. The chosen axioms are sufficiently general to allow the coordinates to belong to any *field:* instead of real numbers we may use rational numbers, complex numbers, or even a finite field such as the residue classes modulo a prime number. Accordingly we speak of the real projective plane, the rational projective plane, the complex projective plane, or a finite projective plane.

14.1 AXIOMS FOR THE GENERAL PROJECTIVE PLANE

> *The more systematic course in the present introductory memoir . . . would have been to ignore altogether the notions of distance and metrical geometry. . . . Metrical geometry is a part of descriptive geometry, and descriptive geometry is all geometry.*
>
> Arthur Cayley *(1821-1895)

The projective plane has already been mentioned in § 6.9. As primitive concepts we take *point, line,* and the relation of *incidence.* If a point and a line are incident, we say that the point lies *on* the line and the line passes *through* the point. The related words *join, meet* (or "intersect"), *concurrent* and *collinear* have their usual meanings. Three non-collinear points are the vertices of a *triangle* whose sides are complete lines. ("Segments" are not defined.) A *complete quadrangle,* its four vertices, its six sides, and its three diagonal points, are defined as in § 1.7. A *hexagon* $A_1B_2C_1A_2B_1C_2$ has six vertices $A_1, B_2, \ldots, C_2$ and six sides

$$A_1B_2,\ B_2C_1,\ C_1A_2,\ A_2B_1,\ B_1C_2,\ C_2A_1.$$

Opposite sides are defined in the obvious manner; for example, A_2B_1 is opposite to A_1B_2. After these preliminary definitions, we are ready for the five axioms.

AXIOM 14.11 *Any two distinct points are incident with just one line.*

NOTATION. The line joining points A and B is denoted by AB.

* *Collected Mathematical Papers,* **2** (Cambridge, 1889), p. 592. Cayley, in 1859, used the word "descriptive" where today we would say "projective." His idea of the supremacy of projective geometry must now be regarded as a slight exaggeration. It is true that projective geometry includes the affine, Euclidean and non-Euclidean geometries; but it does not include the general Riemannian geometry, nor topology.

AXIOM 14.12 *Any two lines are incident with at least one point.*

THEOREM 14.121 *Any two distinct lines are incident with just one point.*

NOTATION. The point of intersection of lines a and b is denoted by $a \cdot b$; that of AB and CD by $AB \cdot CD$. The line joining $a \cdot b$ and $c \cdot d$ is denoted by $(a \cdot b)(c \cdot d)$.

AXIOM 14.13 *There exist four points of which no three are collinear.*

AXIOM 14.14 (Fano's axiom) *The three diagonal points of a complete quadrangle are never collinear.*

AXIOM 14.15 (Pappus's theorem) *If the six vertices of a hexagon lie alternately on two lines, the three points of intersection of pairs of opposite sides are collinear.*

One of the most elegant properties of projective geometry is the *principle of duality,* which asserts (in a projective plane) that every definition remains significant, and every theorem remains true, when we consistently interchange the words *point* and *line* (and consequently interchange *lie on* and *pass through, join* and *intersection, collinear* and *concurrent,* etc.). To establish this principle it will suffice to verify that *the axioms imply their own duals.* Then, given a theorem and its proof, we can immediately assert the dual theorem; for a proof of the latter could be written down mechanically by dualizing every step in the proof of the original theorem.

The dual of Axiom 14.11 is Theorem 14.121, which the reader will have no difficulty in proving (with the help of 14.12). The dual of Axiom 14.12 is one-half of 14.11. The dual of Axiom 14.13 asserts the existence of a *complete quadrilateral,* which is a set of four lines (called *sides*) intersecting in pairs in six distinct points (called *vertices*). Two vertices are said to be *opposite* if they are not joined by a side. The three joins of pairs of opposite vertices are called *diagonals.* If $PQRS$ is a quadrangle with sides

$$p = PQ, \quad q = PS, \quad r = RS, \quad s = QR, \quad w = PR, \quad u = QS,$$

as in Figure 14.1*a*, then $pqrs$ is a quadrilateral with vertices

$$P = p \cdot q, \quad Q = p \cdot s, \quad R = r \cdot s, \quad S = q \cdot r, \quad W = p \cdot r, \quad U = q \cdot s.$$

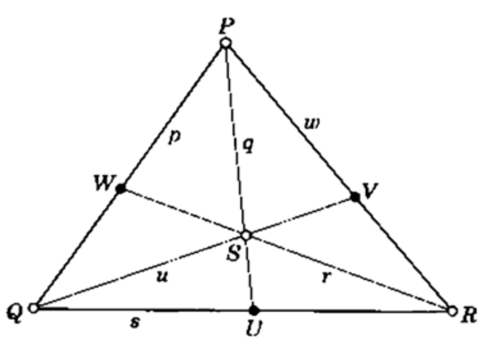


Figure 14.1a

Axiom 14.14 tells us that the three diagonal points

$$U = q \cdot s, \quad V = w \cdot u, \quad W = p \cdot r$$

are not collinear. Its dual asserts that the three diagonals of a complete quadrilateral are never concurrent. If this is false, there must exist a particular quadrilateral whose diagonals are concurrent. Let it be *pqrs*, with diagonals

$$u = QS, \quad v = WU, \quad w = PR.$$

Since these are concurrent, the point $w \cdot u = V$ must lie on v, contradicting the statement that U, V, W are not collinear.

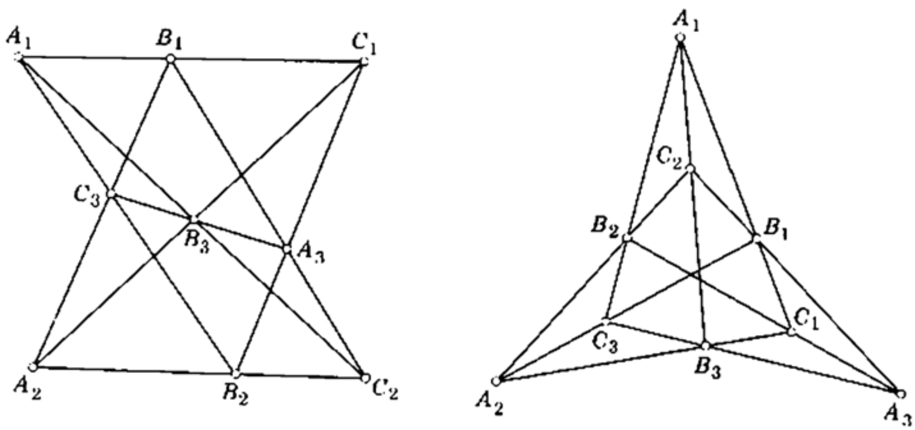


Figure 14.1b

Axiom 14.15 involves nine points and nine lines, which can be drawn in many ways (apparently different though projectively equivalent), such as the two shown in Figure 14.1*b*. $A_1B_2C_1A_2B_1C_2$ is a hexagon whose vertices lie alternately on the two lines $A_1B_1C_1$, $A_2B_2C_2$. The points of intersection of pairs of opposite sides are

$$A_3 = B_1C_2 \cdot B_2C_1, \; B_3 = C_1A_2 \cdot C_2A_1, \; C_3 = A_1B_2 \cdot A_2B_1.$$

The axiom asserts that these three points are collinear. Our notation has been devised in such a way that the three points A_i, B_j, C_k are collinear whenever

$$i + j + k \equiv 0 \qquad (\text{mod } 3).*$$

Another way to express the same result is to arrange the 9 points in the form of a *matrix*

14.151 $$\begin{Vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{Vmatrix}.$$

* Coxeter, Self-dual configurations and regular graphs, *Bulletin of the American Mathematical Society*, **56** (1950), p. 432.

If this were a determinant that we wished to evaluate, we would proceed to multiply the elements in triads. These six "diagonal" triads, as well as the first two rows of the matrix, indicate triads of collinear points. The axiom asserts that the points in the bottom row are likewise collinear. Its inherent self-duality is seen from an analogous matrix of lines

$$\begin{Vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{Vmatrix}.$$

These lines can be picked out in many ways, one of which is

$$\begin{aligned} a_1 &= A_3B_1C_2, & b_1 &= A_1B_3C_2, & c_1 &= A_2B_2C_2, \\ a_2 &= A_2B_3C_1, & b_2 &= A_3B_2C_1, & c_2 &= A_1B_1C_1, \\ a_3 &= A_1B_2C_3, & b_3 &= A_2B_1C_3, & c_3 &= A_3B_3C_3. \end{aligned}$$

This completes our proof of the principle of duality.

EXERCISES

1. Every line is incident with at least three distinct points. (This statement, and the existence of a nonincident point and line, are sometimes used as axioms instead of 14.13 [Robinson **1**, p. 10; Coxeter **2**, p. 13].)

2. A set of m points and n lines is called a *configuration* (m_c, n_d) if c of the n lines pass through each of the points while d of the m points lie on each of the lines. The four numbers are not independent but satisfy $cm = dn$. The dual of (m_c, n_d) is (n_d, m_c).

In the case of a self-dual configuration, we have $m = n$, $c = d$, and the symbol (n_d, n_d) is conveniently abbreviated to n_d. Simple instances are the triangle 3_2, the complete quadrangle $(4_3, 6_2)$ and the complete quadrilateral $(6_2, 4_3)$. Axiom 14.14 asserts the nonexistence of the *Fano configuration** 7_3. The points and lines that occur in Axiom 14.15 (Figure 14.1*b*) form the *Pappus configuration* 9_3, which may be regarded (in how many ways?) as a cycle of three triangles such as

$$A_1B_1C_2, \quad A_2B_2C_3, \quad A_3B_3C_1,$$

each inscribed in the next (cf. Figure 1.8*a*, where UVW is inscribed in ABC). The self-duality is evident.

By a suitable change of notation, Axiom 14.15 may be expressed thus: *If AB, CD, EF are concurrent, and DE, FA, BC are concurrent, then AD, BE, CF are concurrent.*

3. A particular *finite projective plane,* in which only 13 "points" and 13 "lines" exist, can be defined abstractly by calling the points P_i and the lines p_i $(i = 0, 1, \ldots, 12)$ with the rule that P_i and p_j are "incident" if and only if

$$i + j \equiv 0, 1, 3 \text{ or } 9 \pmod{13}.$$

Construct a table to indicate the 4 points on each line and the 4 lines through each point [Veblen and Young **1**, p. 6]. Verify that all the axioms are satisfied; for example, $P_0P_1P_2P_5$ is a complete quadrangle with sides

$$P_0P_1 = p_0, \quad P_0P_2 = p_1, \quad P_1P_5 = p_8, \quad P_0P_5 = p_9, \quad P_2P_5 = p_{11}, \quad P_1P_2 = p_{12}$$

* Coxeter, *Bulletin of the American Mathematical Society,* **56** (1950), pp. 423–425.

and diagonal points $P_3 = p_0 \cdot p_{11}$, $P_4 = p_9 \cdot p_{12}$, $P_8 = p_1 \cdot p_8$. A possible matrix for Axiom 14.15 is

$$\left\| \begin{matrix} P_0 & P_2 & P_8 \\ P_3 & P_4 & P_6 \\ P_9 & P_{10} & P_5 \end{matrix} \right\|.$$

The first row may be any set of three collinear points. The second row may be any such set on a line not incident with a point in the first row. The last row is then determined; e.g., in the above instance it consists of

$$P_2P_6 \cdot P_4P_8 = P_9, \qquad P_3P_8 \cdot P_0P_6 = P_{10}, \qquad P_0P_4 \cdot P_2P_3 = P_5.$$

This differs from the general "Pappus matrix" 14.151 in that sets of collinear points occur not only in the rows and generalized diagonals but also in the columns. In other words, the 9 points form a configuration which is not merely 9_3 but $(9_4, 12_3)$. When any one of the 9 points is omitted, the remaining 8 form a self-dual configuration 8_3 which may be regarded as a pair of mutually inscribed quadrangles (such as $P_0P_9P_5P_8$ and $P_2P_3P_{10}P_6$). [Hilbert and Cohn-Vossen **1**, pp. 101–102.]

4. The geometry described in Ex. 3 is known as $PG(2, 3)$. More generally, $PG(2, p)$ is a finite plane in which each line contains $p+1$ points. Consequently, each point lies on $p+1$ lines. There are p^2+p+1 points (and the same number of lines) altogether. In other words, the whole geometry is a configuration n_d with $n = p^2+p+1$ and $d = p+1$. (Actually p is not arbitrary, e.g., although it may be any power of an odd prime, for instance, 5, 7, or 9, it cannot be 6.)* The possibility of such finite planes indicates that the projective geometry defined by Axioms 14.11 to 14.15 is not *categorical:* it is not just one geometry but many geometries, in fact, infinitely many.

5. In any finite projective geometry, Sylvester's theorem (§ 4.7) is false.

14.2 PROJECTIVE COORDINATES

Modern algebra does not seem quite so terrifying when expressed in these geometrical terms!

G. de B. Robinson (1906-)

[Robinson **1**, p. 94]

We saw, in § 13.7, that three real numbers t_1, t_2, t_3 will serve as barycentric coordinates for a point in the affine plane (with respect to any given triangle of reference) if and only if

$$t_1 + t_2 + t_3 \neq 0.$$

Also a linear homogeneous equation 13.72 will serve as the equation for a line if and only if the coefficients T_1, T_2, T_3 are not all equal. The remarks

* By not insisting on Axiom 14.14, we can develop a "geometry of characteristic 2" in which p is a power of 2. By not insisting on Axiom 14.15, we can develop a "non-Desarguesian plane." For the application to mutually orthogonal Latin squares, see Robinson **1**, p. 161, Appendix II.

just after 13.84 indicate that these artificial restrictions will be avoided when we have extended the real affine plane to the real projective plane by adding the line at infinity

14.21 $$t_1 + t_2 + t_3 = 0$$

and all its points (which are the points at infinity in various directions).

When we interpret T_1, T_2, T_3 as the distances from A_1, A_2, A_3 to the line

$$T_1 t_1 + T_2 t_2 + T_3 t_3 = 0,$$

it is obvious that a parallel line is obtained by adding the same number to all three T's. Hence the point of intersection of two parallel lines satisfies 14.21, that is, it lies on the line at infinity.

To emphasize the fact that, in projective geometry, the line at infinity no longer plays a special role, we shall abandon the barycentric coordinates (t_1, t_2, t_3) in favor of general *projective* coordinates (x_1, x_2, x_3), given by

$$t_1 = \mu_1 x_1, \quad t_2 = \mu_2 x_2, \quad t_3 = \mu_3 x_3,$$

where μ_1, μ_2, μ_3 are constants, $\mu_1\mu_2\mu_3 \neq 0$. Thus (x_1, x_2, x_3) is the centroid of masses $\mu_\alpha x_\alpha$ at A_α $(\alpha = 1, 2, 3)$, and the line at infinity has the undistinguished equation

$$\mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3 = 0.$$

The contrast between these two kinds of coordinates may also be expressed as follows. Barycentric coordinates can be referred to any given triangle; the "simplest" points

$$(1, 0, 0), \quad (0, 1, 0), \quad (0, 0, 1)$$

are the vertices, and the *unit point* (1, 1, 1) is the centroid. More usefully, projective coordinates can be referred to any given quadrangle! Taking three of the four vertices to determine a system of barycentric coordinates, suppose the fourth vertex is (μ_1, μ_2, μ_3). By using these μ's for the transition to projective coordinates, we give this fourth vertex the new coordinates (1, 1, 1). Just as, in affine geometry, all triangles are alike, so in projective geometry *all quadrangles are alike.*

To prove that projective coordinates provide a model (in the augmented affine plane) for the abstract projective plane described in § 14.1, we can take each of our geometric axioms and prove it analytically (i.e., algebraically).

To prove 14.11, we merely have to observe that the line joining points (y_1, y_2, y_3) and (z_1, z_2, z_3) is

14.22 $$\begin{vmatrix} y_2 & y_3 \\ z_2 & z_3 \end{vmatrix} x_1 + \begin{vmatrix} y_3 & y_1 \\ z_3 & z_1 \end{vmatrix} x_2 + \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} x_3 = 0$$

(cf. 13.74). Similarly, for 14.12 (or rather, 14.121), the point of intersection of lines $\Sigma Y_\alpha x_\alpha = 0$ and $\Sigma Z_\alpha x_\alpha = 0$ is

$$\left(\begin{vmatrix} Y_2 & Y_3 \\ Z_2 & Z_3 \end{vmatrix}, \begin{vmatrix} Y_3 & Y_1 \\ Z_3 & Z_1 \end{vmatrix}, \begin{vmatrix} Y_1 & Y_2 \\ Z_1 & Z_2 \end{vmatrix}\right).$$

For 14.13, we can use the four points

14.23 $$(1, 0, 0), \quad (0, 1, 0), \quad (0, 0, 1), \quad (1, 1, 1).$$

The diagonal points of the quadrangle so formed are

$$(0, 1, 1), \quad (1, 0, 1), \quad (1, 1, 0).$$

If these three points lay on a line $\Sigma X_\alpha x_\alpha = 0$, we should have

14.24 $$X_2 + X_3 = 0, \quad X_3 + X_1 = 0, \quad X_1 + X_2 = 0,$$

whence $X_1 = X_2 = X_3 = 0$, which is absurd. This proves 14.14.

Finally, to prove 14.15 we use the coordinates 14.23 for the four points

$$A_1, \quad A_2, \quad A_3, \quad C_1.$$

On the lines C_1A_1, C_1A_2, C_1A_3, which are

$$x_2 = x_3, \quad x_3 = x_1, \quad x_1 = x_2,$$

we take the points B_1, B_3, B_2 to be

$$(p, 1, 1), \quad (1, q, 1), \quad (1, 1, r).$$

The three lines A_3B_1, A_1B_3, A_2B_2, being

$$x_1 = px_2, \quad x_2 = qx_3, \quad x_3 = rx_1,$$

all pass through the same point C_2 if

14.25 $$pqr = 1.$$

The three lines A_3B_3, A_2B_1, A_1B_2, being

$$x_2 = qx_1, \quad x_1 = px_3, \quad x_3 = rx_2,$$

all pass through the same point C_3 if

$$qpr = 1.$$

Since this condition agrees with 14.25, the proof is complete. However, it is important to observe that the above deduction can be carried through in the more general situation where the coordinates belong not to a *field* but to an arbitrary *division ring* [Birkhoff and MacLane **1**, p. 222]. We can still speak of points and lines, but Axiom 14.15 will have to be replaced by a weaker statement if the coordinate ring includes elements p and q such that

$$pq \neq qp.$$

For instance, we might have $p = k$ and $q = j$ in a "quaternion geometry" whose coordinates are based on "units" i, j, k satisfying

$$i^2 = j^2 = k^2 = ijk = -1.$$

When the A's and B's are so chosen, 14.15 is false. We have thus established an important connection between geometry and algebra: Hilbert's discovery that, when homogeneous coordinates are used in a plane satisfying the first four axioms, *Pappus's theorem is equivalent to the commutative law for multiplication.*

EXERCISES

1. Given five points, no three collinear, we can assign the coordinates 14.23 to any four of them, and then the coordinates (x_1, x_2, x_3) of the fifth are definite (apart from the possibility of multiplying all by the same constant). If the mutual ratios of the three x's are rational, we can multiply by a "common denominator" so as to make them all integral. In this case we can derive the fifth point from the first four by a linear construction, involving a finite sequence of operations of joining two known points or taking the point of intersection of two known lines. Devise such a construction for the point $(1, 2, 3)$.

2. The four points $(1, \pm 1, \pm 1)$ form a complete quadrangle whose diagonal triangle is the triangle of reference.

3. A configuration 8_3, consisting of two mutually inscribed quadrangles, exists in the complex projective plane, but not in the real projective plane. When it does exist, its eight points appear in four pairs of "opposites" whose joins are concurrent. The complete figure is a $(9_4, 12_3)$. *Hint*: Let the two quadrangles be $P_0P_2P_4P_6$ and $P_1P_3P_5P_7$, so that the sets of three collinear points are

$$P_0P_1P_3,\ P_1P_2P_4,\ P_2P_3P_5,\ P_3P_4P_6,\ P_4P_5P_7,\ P_5P_6P_0,\ P_6P_7P_1,\ P_7P_0P_2.$$

Take $P_0P_1P_2$ as triangle of reference and let P_3, P_4, P_7 be $(1, 1, 0)$, $(0, 1, 1)$, $(1, 0, x)$. Deduce that P_5 and P_6 are $(1, 1, x+1)$ and $(1, x+1, x)$. Obtain an equation for x from the collinearity of $P_0P_5P_6$.

4. If p is an odd prime, a finite projective plane $PG(2, p)$ can be obtained by taking the coordinates to belong to the field $GF(p)$ which consists of the p residues (or, strictly, residue classes) modulo p [Ball **1**, pp. 60–61]. For instance, the appropriate "finite arithmetic" for $PG(2, 3)$ consists of symbols 0, 1, 2 which behave like ordinary integers except that

$$1 + 2 = 0 \quad \text{and} \quad 2 \times 2 = 1.$$

In the notation of Ex. 3 at the end of § 14.1, take $P_0P_1P_2$ to be the triangle of reference and P_5 the unit point $(1, 1, 1)$. Find coordinates for the remaining points, and equations for the lines.

Finite planes, and the analogous finite n-spaces $PG(n, p)$, were discovered by von Staudt* and rediscovered by Fano. Von Staudt took n to be 2 or 3. Fano took p to be a prime. The generalization $PG(n, p^k)$ is credited to Veblen and Bussey.

5. Taking the coordinates to belong to $GF(2)$, which consists of the two "numbers" 0 and 1 with the rule for addition

$$1 + 1 = 0,$$

we obtain a finite "geometry" in which the diagonal points of a complete quadrangle are always collinear! Our proof of 14.14 breaks down because now the equations 14.24

* K. G. C. von Staudt, *Beiträge zur Geometrie der Lage*, vol. I (Nürnberg, 1856), pp. 87–88; Gino Fano, *Giornale di Matematiche*, **30** (1892), pp. 114–124; Veblen and Bussey, *Transactions of the American Mathematical Society*, **7** (1906), pp. 241–259.

have not only the inadmissible solution $X_1 = X_2 = X_3 = 0$ but also the significant solution $X_1 = X_2 = X_3 = 1$, which yields the line

$$x_1 + x_2 + x_3 = 0.$$

This $PG(2, 2)$ can be described abstractly by calling its seven points P_i and its seven lines p_i ($i = 0, 1, \ldots, 6$) with the rule that P_i and p_j are incident if and only if

$$i + j \equiv 0, 1 \text{ or } 3 \pmod 7.$$

14.3 DESARGUES'S THEOREM

The fundamental idea for this pure geometry came from the desire of Renaissance painters to produce a "visual" geometry. How do things really look, and how can they be presented on the plane of the drawing? For example, there will be no parallel lines, since such lines appear to the eye to converge.

S. H. Gould (1909-)
[Gould **1**, p. 298]

Two triangles, with their vertices named in a particular order, are said to be *perspective from a point* (or briefly, "perspective") if their three pairs of corresponding vertices are joined by concurrent lines. For instance, in Figure 14.1*b*, the triangles $A_1A_2A_3$ and $B_1B_3B_2$ (*sic*) are perspective from C_1. By permuting the vertices of $B_1B_3B_2$ cyclically, either forwards or backwards, we see that the same two triangles are also perspective from C_2 or C_3. In fact, one of the neatest statements of Axiom 14.15 [see Veblen and Young **1**, p. 100] is:

If two triangles are doubly perspective they are trebly perspective.

Dually, two triangles are said to be *perspective from a line* if their three pairs of corresponding sides meet in collinear points. It was observed by G. Hessenberg* that our axioms suffice for a proof of

DESARGUES'S THEOREM. *If two triangles are perspective from a point they are perspective from a line, and conversely.*

The details are as follows. Let two triangles PQR and $P'Q'R'$ be perspective from O, as in Figure 14.3*a*, and let their corresponding sides meet in points

$$D = QR \cdot Q'R', \quad E = RP \cdot R'P', \quad F = PQ \cdot P'Q'.$$

We wish to prove that D, E, F are collinear. After defining four further points

$$S = PR \cdot Q'R', \qquad T = PQ' \cdot OR,$$
$$U = PQ \cdot OS, \qquad V = P'Q' \cdot OS,$$

we have, in general,† enough triads of collinear points to make three applications of Axiom 14.15. The "matrix" notation enables us to write simply

* *Mathematische Annalen*, **61** (1905), pp. 161-172.

$$\begin{Vmatrix} O & Q & Q' \\ P & S & R \\ D & T & U \end{Vmatrix}, \quad \begin{Vmatrix} O & P & P' \\ Q' & R' & S \\ E & V & T \end{Vmatrix}, \quad \begin{Vmatrix} P & Q' & T \\ V & U & S \\ D & E & F \end{Vmatrix}.$$

The last row of the last matrix exhibits the desired collinearity.

The converse follows by the principle of duality.

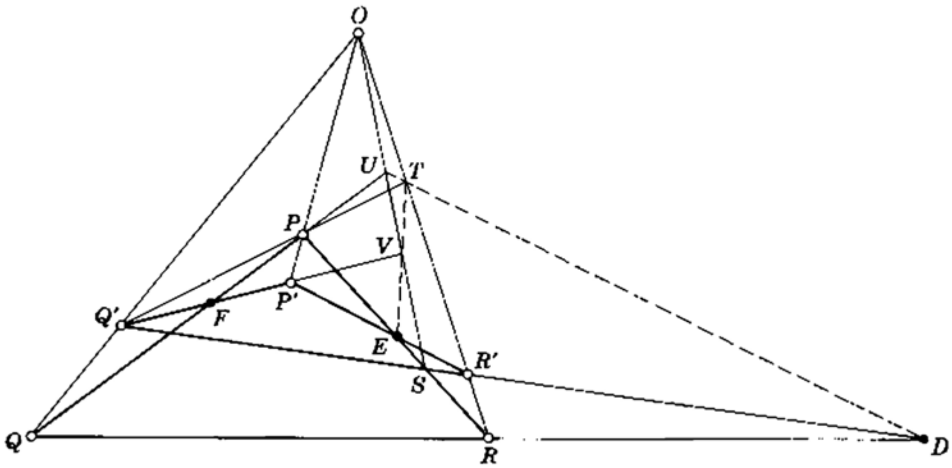


Figure 14.3a

EXERCISES

1. The triangle $(p, 1, 1)$ $(1, q, 1)$ $(1, 1, r)$ is perspective with the triangle of reference from the unit point $(1, 1, 1)$. Pairs of corresponding sides meet in the three collinear points

$$(0, q - 1, 1 - r), \quad (1 - p, 0, r - 1), \quad (p - 1, 1 - q, 0).$$

2. Desargues's theorem involves 10 points and 10 lines, forming a configuration 10_3. To obtain a symmetrical notation, consider triangles $P_{14}P_{24}P_{34}$ and $P_{15}P_{25}P_{35}$, perspective from a point P_{45} and consequently from a line $P_{23}P_{31}P_{12}$. Then three points P_{ij} are collinear if their subscripts involve just three of the numbers 1, 2, 3, 4, 5. If the remaining two of the five numbers are k and l, we may call the line p_{kl}. Then the same two triangles may be described as $p_{15}\,p_{25}\,p_{35}$ and $p_{14}\,p_{24}\,p_{34}$, perspective from the line p_{45}.

3. In the finite projective plane $PG(2, 3)$, the two triangles $P_1P_2P_7$ and $P_3P_8P_4$ are perspective from the point P_0 and from the line $P_9P_{12}P_{10}$. Identify the remaining points in Figure 14.3*a*. (In this special geometry, U and V both coincide with F, which is not surprising in view of the fact that Figure 14.3*a* involves 14 points whereas the whole plane contains only 13.)

14.4 QUADRANGULAR AND HARMONIC SETS

Desargues's theorem enables us to prove an important property of a

quadrangular set of points, which is the section of the six sides of a complete quadrangle by any line that does not pass through a vertex:

14.41 *Each point of a quadrangular set is uniquely determined by the remaining points.*

Proof. Let $PQRS$ be a complete quadrangle whose sides PS, QS, RS, QR, RP, PQ meet a line g (not through a vertex) in six points A, B, C, D, E, F, certain pairs of which may possibly coincide. (The first three points come from three sides all containing the same vertex S; the last three from the respectively opposite sides, which form the triangle PQR.) To show that F is uniquely determined by the remaining five points, we set up another quadrangle $P'Q'R'S'$ whose first five sides pass through A, B, C, D, E, as in Figure 14.4*a*. Since the two triangles PRS and $P'R'S'$ are perspective from the line g, the converse of Desargues's theorem tells us that they are also perspective from a point; thus PP' passes through the point $O = RR' \cdot SS'$. Similarly, the perspective triangles QRS and $Q'R'S'$ show that QQ' passes through this same point O. In fact, all the four lines PP' QQ', RR', SS' pass through O, so that $PQRS$ and $P'Q'R'S'$ are "persp tive quadrangles." By the direct form of Desargues's theorem, the triang PQR and $P'Q'R'$, which are perspective from the point O, are also persp tive from the line DE, which is g; that is, the sides PQ and $P'Q'$ both m g in the same point F.

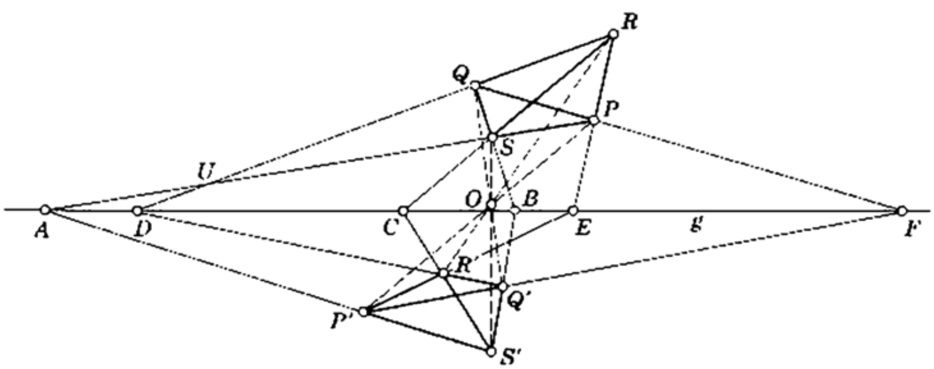


Figure 14.4a

We shall find it convenient to use the symbol

$$(AD)\ (BE)\ (CF)$$

to denote the statement that the six points form a quadrangular set in the above manner. This statement is evidently unchanged if we apply any permutation to ABC and the same permutation to DEF. It is also equivalent to any of

$$(AD)\ (EB)\ (FC),\quad (DA)\ (BE)\ (FC),\quad (DA)\ (EB)\ (CF).$$

To obtain other permutations we need a new quadrangle. With the ex-

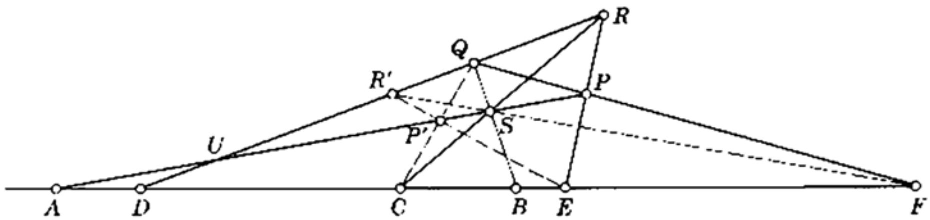


Figure 14.4b

ercise of some ingenuity we can retain two of the four old vertices, say Q and S. Defining

$$R' = QR \cdot SF, \quad P' = PS \cdot QC,$$

as in Figure 14.4*b*, we apply Axiom 14.15 to the hexagon $PRQCFS$ according to the scheme

$$\left\| \begin{matrix} P & F & Q \\ C & R & S \\ R' & P' & E \end{matrix} \right\|,$$

with the conclusion that $R'P'$ passes through E. Just as the quadrangle $PQRS$ yields $(AD)\,(BE)\,(CF)$, the quadrangle $P'QR'S$ yields $(AD)\,(BE)\,(FC)$. In other words, the statement $(AD)\,(BE)\,(CF)$ implies $(AD)\,(BE)\,(FC)$, and hence also

14.42 $(AD)\,(BE)\,(CF)$ *implies* $(DA)\,(EB)\,(FC)$.

In the important special case $(AA)\,(BB)\,(CF)$, which is abbreviated to

$$\mathrm{H}(AB, CF),$$

we say that the four points form a *harmonic set*, or, more precisely, that F is the *harmonic conjugate* of C with respect to A and B. This means that A and B are two of the three diagonal points of a quadrangle while C and F lie respectively on the remaining sides, that is, on the sides that pass through the third diagonal point. Axiom 14.14 tells us that the harmonic conjugates C and F are distinct (except in the degenerate case when they coincide with A or B).

EXERCISES

1. H(AB, CF) is equivalent to H(BA, CF) or H(AB, FC) or H(BA, FC).

2. Describe in detail a construction for the harmonic conjugate of C with respect to two given points A and B (on a line through C, as in Figure 14.4c).

3. The harmonic conjugate of $(0, 1, \lambda)$ with respect to $(0, 1, 0)$ and $(0, 0, 1)$ is $(0, 1, -\lambda)$.

4. In $PG(2, 3)$ (see Ex. 3 at the end of § 14.1), every set of four collinear points is a harmonic set in every order; e.g., H(P_0P_1, P_3P_9), H(P_0P_3, P_9P_1), H(P_0P_9, P_1P_3).

5. In Figure 6.6a, H(AA', A_1A_2). Deduce the metrical definition

$$\frac{AA_1}{A_1A'} = \frac{AA_2}{A'A_2}$$

for a harmonic set. (*Hint*: Defining E' as in Ex. 4 at the end of § 6.6, consider the quadrangle formed by P, E, E' and the point at infinity on A_1P.)

14.5 PROJECTIVITIES

A *range* is the set of all points on a line. Dually, a *pencil* is the set of all lines through a point. Ranges and pencils are instances of *one-dimensional forms*. We shall often have occasion to consider a (one-to-one) correspondence between two one-dimensional forms. The simplest possible correspondence between a range and a pencil arises when the lines of the pencil join the points of the range to another point, so that the range is a *section* of the pencil. The correspondence between two ranges that are sections of one pencil by two distinct lines is called a *perspectivity*; in such a case we write

$$X \,\overset{}{\underset{\wedge}{=}}\, X' \quad \text{or} \quad X \,\overset{O}{\underset{\wedge}{=}}\, X',$$

meaning that, if X and X' are corresponding points of the two ranges, their join XX' continually passes through a fixed point O, which we call the *center* of the perspectivity. There is naturally also a dual kind of perspectivity relating pencils instead of ranges.

The product of any number of perspectivities is called a *projectivity*. Two ranges (or pencils) related by a projectivity are said to be *projectively related*, and we write

$$X \,\underset{\wedge}{-}\, X'.$$

For instance, in the circumstances illustrated in Figure 14.5a,

$$ABCD \,\overset{O}{\underset{\wedge}{=}}\, A_0B_0C_0D_0 \,\overset{O_1}{\underset{\wedge}{=}}\, A'B'C'D', \qquad ABCD \,\underset{\wedge}{-}\, A'B'C'D'.$$

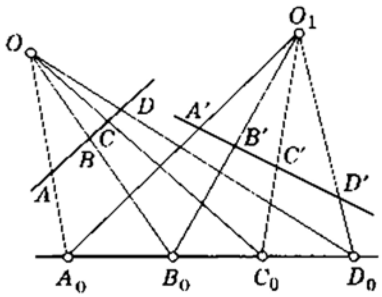


Figure 14.5a

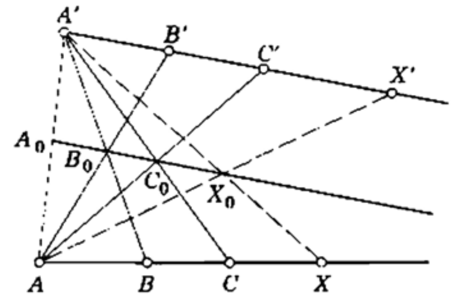


Figure 14.5b

Analogously, we can define a projectivity relating a range to a pencil, or vice versa.

Given three distinct points A, B, C on a line, and three distinct points A', B', C' on another line, we can relate them by a pair of perspectivities in the manner of Figure 14.5*b*, where the *axis* (or "intermediary line") of the projectivity joins the points

$$B_0 = AB' \cdot BA', \quad C_0 = AC' \cdot CA',$$

so that

$$ABC \overset{A'}{\doublebarwedge} A_0B_0C_0 \overset{A}{\doublebarwedge} A'B'C'.$$

For each point X on AB we obtain a corresponding point X' on $A'B'$ by joining A to the point $X_0 = A'X \cdot B_0C_0$, so that

$$ABCX \overset{A'}{\doublebarwedge} A_0B_0C_0X_0 \overset{A}{\doublebarwedge} A'B'C'X'.$$

By Axiom 14.15, the axis B_0C_0, being the "Pappus line" of the hexagon $AB'CA'BC'$, contains the point $BC' \cdot CB'$. Similarly, it contains the point of intersection of the "cross joins" of any two pairs of corresponding points. In particular, we could have derived the same point X' from a given point X by using perspectivities from B' and B (or any other pair of corresponding points) instead of A' and A.

It can be proved [Baker **1**, pp. 62–64; Robinson **1**, pp. 24–36] that the product of any number of perspectivities can be reduced to such a product of two, provided the initial and final ranges are not on the same line. In other words,

14.51 *Any projectivity relating ranges on two distinct lines is expressible as the product of two perspectivities whose centers are corresponding points (in reversed order) of the two related ranges.*

To relate two triads of distinct points ABC and $A'B'C'$ on one line, we may use an arbitrary perspectivity $ABC \doublebarwedge A_1B_1C_1$ to obtain a triad on another line, and then relate $A_1B_1C_1$ to $A'B'C'$ as in 14.51. Hence

14.52 *It is possible, by a sequence of not more than three perspectivities, to relate any three distinct collinear points to any other three distinct collinear points.*

A projectivity $X \barwedge X'$ on one line may have one or more *invariant* points (such that $X = X'$). If it has more than two invariant points, it is merely the identity, $X \barwedge X$. In fact, the above construction for a projectivity

$$ABCX \barwedge ABCX'$$

on one line involves four points on another line such that

$$ABCX \doublebarwedge A_1B_1C_1X_1 \barwedge ABCX'.$$

By 14.51, there is essentially only one projectivity $A_1B_1C_1 \barwedge ABC$. We have thus proved

THE FUNDAMENTAL THEOREM OF PROJECTIVE GEOMETRY. *A projectivity is determined when three points of one range and the corresponding three points of the other are given.*

If a projectivity relating ranges on two distinct lines has an invariant point A, this point, belonging to both ranges, must be the common point of the two lines, as in Figure 14.5*c*. Let B and C be any other points of the first range, B' and C' the corresponding points of the second. The fundamental theorem tells us that the perspectivity

$$ABC \overset{O}{\doublebarwedge} AB'C',$$

where $O = BB' \cdot CC'$, is the same as the given projectivity $ABC \barwedge AB'C'$. Hence

14.53 *A projectivity between two distinct lines is equivalent to a perspectivity if and only if their point of intersection is invariant.*

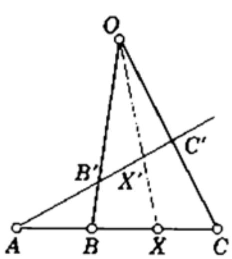


Figure 14.5c

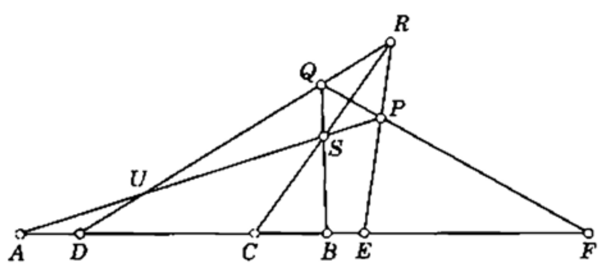


Figure 14.5d

Returning to the notion of a projectivity between ranges on one line (i.e., a projective transformation of the line into itself), we recall that, if such a transformation is not merely the identity, it cannot have more than two invariant points. It is said to be *elliptic, parabolic,* or *hyperbolic* according as the number of invariant points is 0, 1, or 2. When coordinates are used,

invariant points arise from roots of quadratic equations; thus elliptic projectivities do not occur in complex geometry, but

$$ABC \barwedge BCA$$

is elliptic in real geometry [Coxeter **2**, p. 48].

Figure 14.5*d* (cf. 14.4*a*) suggests a simple construction for a hyperbolic projectivity $ABF \barwedge ACE$ in which one of the invariant points is given:

$$ABF \overset{Q}{\doublebarwedge} ASP \overset{R}{\doublebarwedge} ACE.$$

Here S and P may be any two points collinear with A, and then the other two vertices of the quadrangle are

$$Q = BS \cdot FP, \quad R = CS \cdot EP.$$

The second invariant point is evidently D, on QR. When the same projectivity is expressed in the form $ADB \barwedge ADC$ (that is, when both invariant points are given), we have the analogous construction

$$ADB \overset{Q}{\doublebarwedge} AUS \overset{R}{\doublebarwedge} ADC,$$

where $U = AS \cdot QD$. This can still be carried out if A and D coincide (i.e., if g passes through the diagonal point $U = PS \cdot QR$ of the quadrangle), in which case we have the parabolic projectivity

$$AAB \barwedge AAC$$

[Coxeter **2**, p. 50].

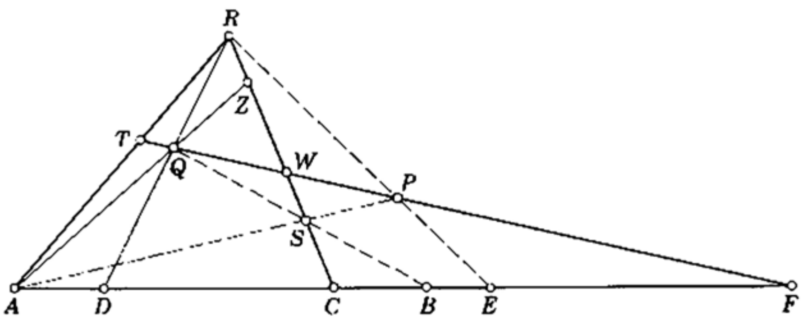


Figure 14.5e

An *involution* is a projectivity of period 2, that is, a projectivity which *interchanges* pairs of points. Figure 14.5*e* is derived from Figure 14.5*d* by adding extra points T, W, Z. We may imagine this figure to have been derived from *any* four given collinear points A, C, D, F by taking a point R outside their line, letting the joins RA, RD, RC meet an arbitrary line through F in T, Q, W, respectively, and then taking $Z = AQ \cdot RC$. Since

$$ADCF \overset{Q}{\underset{\wedge}{=}} ZRCW \overset{A}{\underset{\wedge}{=}} QTFW \overset{R}{\underset{\wedge}{=}} DAFC,$$

we have

14.54 $$ADCF \underset{\wedge}{-} DAFC.$$

But, by the fundamental theorem, there is only one projectivity $ADC \underset{\wedge}{-} DAF$. Hence, if a projectivity interchanging A and D transforms C into F, it *interchanges* C and F. In other words,

14.55 *Any projectivity that interchanges two points is an involution.*

Applying the same set of three perspectivities to another point B, we have

$$B \overset{Q}{\underset{\wedge}{=}} S \overset{A}{\underset{\wedge}{=}} P \overset{R}{\underset{\wedge}{=}} E.$$

Since $\mathbf{Q}(ABC, DEF)$, we have now proved the theorem of the quadrangular set:

14.56 *The three pairs of opposite sides of a quadrangle meet any line (not through a vertex) in three pairs of an involution.*

Combining this with 14.55, we have an alternative proof for 14.42 [Veblen and Young **1**, p. 101].

Since the involution $ACD \underset{\wedge}{-} DFA$ is determined by its pairs AD and CF (or any other two of its pairs), it is conveniently denoted by

$$(AD)(CF)$$

or $(DA)(CF)$ or $(CF)(AD)$, etc. Thus $(AD)(BE)(CF)$ implies that the pair BE belongs to $(AD)(CF)$, and CF to $(AD)(BE)$, and AD to $(BE)(CF)$. The points in a pair are not necessarily distinct. When $A = D$ and $B = E$, so that $\mathrm{H}(AB, CF)$, we have the hyperbolic involution $(AA)(BB)$ which interchanges pairs of harmonic conjugates with respect to A and B. Since this same involution is expressible as $(AA)(CF)$,

14.57 *If an involution has one invariant point, it has another, and consists of the correspondence between harmonic conjugates with respect to these two points.*

It follows that there is no parabolic involution.

EXERCISES

1. Let the lines $OA, OB, \ldots, O_1A', O_1B', \ldots$ and A_0B_0 in Figure 14.5*a* be denoted by $a, b, \ldots, a', b', \ldots$ and o. Use the principle of duality to justify the notation

$$abcd \overset{o}{\underset{\wedge}{=}} a'b'c'd'.$$

2. The harmonic property is invariant under a projectivity: if $\mathrm{H}(AB, CF)$ and $ABCF \underset{\wedge}{-} A'B'C'F'$, then $\mathrm{H}(A'B', C'F')$ [Coxeter **2**, p. 23].

3. $\mathrm{H}(AB, CF)$ implies $\mathrm{H}(CF, AB)$. (*Hint*: By 14.54, $ACBF \underset{\wedge}{-} CAFB$.)

4. Draw a quadrangle and its section, as in Figure 14.5*d*. Take an arbitrary point X on g and construct the corresponding point X' in the hyperbolic projectivity

$$ABF \barwedge ACE.$$

Do the same for $ADB \barwedge ADC$, and draw the modified figure that is appropriate for the parabolic projectivity $AAB \barwedge AAC$.

5. Two perspectivities cannot suffice for the construction of an elliptic projectivity.

6. In the notation of Figure 14.4*b*,

$$ADCF \overset{Q}{\doublebarwedge} AUP'P \overset{E}{\doublebarwedge} DUR'R \overset{S}{\doublebarwedge} DAFC.$$

7. Any projectivity may be expressed as the product of two involutions [Coxeter **2**, p. 54].

8. The projectivities on the line $x_3 = 0$ are the linear transformations

$$\mu x'_1 = c_{11}x_1 + c_{12}x_2,$$

$$\mu x'_2 = c_{21}x_1 + c_{22}x_2,$$

where $c_{11}c_{22} \neq c_{12}c_{21}$. Under what circumstances is such a projectivity (i) parabolic, (ii) an involution?

14.6 COLLINEATIONS AND CORRELATIONS

A *collineation* is a transformation (of the plane) which transforms collinear points into collinear points. Thus it transforms lines into lines, ranges into ranges, pencils into pencils, quadrangles into quadrangles, and so on. A *projective collineation* is a collineation which transforms every one-dimensional form projectively.

14.61 *Any collineation that transforms one range into a projectively related range is a projective collineation.*

Proof [Bachmann **1**, p. 85]. Let the given collineation transform the range of points X on a certain line a into a projectively related range of points X' on the corresponding line a', and let it transform the points Y on another line b into corresponding points Y' on b'. Any perspectivity relating X and Y will be transformed into a perspectivity relating X' and Y'. Hence

$$Y \doublebarwedge X \barwedge X' \doublebarwedge Y',$$

so that the collineation induces a projectivity $Y \barwedge Y'$ between the points of b and b', as desired.

It follows that a projective collineation is determined when two corresponding quadrangles (or quadrilaterals) are given [Coxeter **2**, p. 60].

A *perspective collineation* with center O and axis o is a collineation which leaves invariant all the lines through O and all the points on o. (By 14.61, every perspective collineation is a projective collineation.) Following Sophus Lie (1842–1899), we call a perspective collineation an *elation* or a *homology*

according as the center and axis are or are not incident. A *harmonic* homology is the special case when corresponding points A and A', on a line a through O, are harmonic conjugates with respect to O and $o \cdot a$. Every projective collineation of period 2 is a harmonic homology [Coxeter **2**, p. 64].

We have seen that a collineation is a point-to-point and line-to-line transformation which preserves incidences. Somewhat analogously, a *correlation* is a point-to-line and line-to-point transformation which dualizes incidences: it transforms points A into lines a', and lines b into points B', in such a way that a' passes through B' if and only if A lies on b. Thus a correlation transforms collinear points into concurrent lines (and vice versa), ranges into pencils, quadrangles into quadrilaterals, and so on. A *projective correlation* is a correlation that transforms every one-dimensional form projectively. In a manner resembling the proof of 14.61, we can establish

14.62 *Any correlation that transforms one range into a projectively related pencil (or vice versa) is a projective correlation.*

It follows that a projective correlation is determined when a quadrangle and the corresponding quadrilateral are given [Coxeter **2**, p. 66].

A *polarity* is a projective correlation of period 2. In general, a correlation transforms a point A into a line a' and transforms this line into a new point A''. When the correlation is of period two, A'' always coincides with A and we can simplify the notation by omitting the prime (′). Thus a polarity relates A to a, and vice versa. Following J. D. Gergonne (1771–1859), we call a the *polar* of A, and A the *pole* of a. Clearly, the polars of all the points on a form a projectively related pencil of lines through A.

Since a polarity dualizes incidences, if A lies on b, a passes through B. In this case we say that A and B are *conjugate points*, a and b are *conjugate lines*. It may happen that A and a are incident, so that each is *self-conjugate*. We can be sure that this does not always happen, for it is easy to prove that the join of two self-conjugate points cannot be a self-conjugate line. It is slightly harder to prove that no line can contain more than two self-conjugate points [Coxeter **2**, p. 68]. The following theorem will be used in § 14.7:

14.63 *A polarity induces an involution of conjugate points on any line that is not self-conjugate.*

Proof. On a non-self-conjugate line a, the projectivity $X \barwedge a \cdot x$ (Figure 14.6a) transforms any non-self-conjugate point B into another point $C = a \cdot b$, whose polar is AB. The same projectivity transforms C into B. Since it interchanges B and C, it must be an involution.

Dually, x and AX are paired in the involution of conjugate lines through A.

Such a triangle ABC, in which each vertex is the pole of the opposite side (so that any two vertices are conjugate points, and any two sides are conjugate lines), is said to be *self-polar*. If P is any point not on a side, its

polar p does not pass through a vertex, and the polarity may be described as the unique projective correlation that transforms the quadrangle $ABCP$ into the quadrilateral $abcp$. An appropriate symbol, analogous to the symbol $(AB)(PQ)$ for an involution, is

$$(ABC)(Pp).$$

Thus any triangle ABC, any point P not on a side, and any line p not through a vertex, determine a definite polarity $(ABC)(Pp)$, in which the polar x of an arbitrary point X can be constructed by simple incidences. As a first step towards this construction we need the following lemma:*

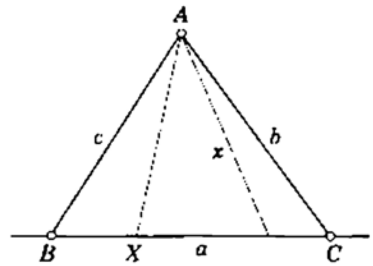


Figure 14.6a

14.64 *If the polars of the vertices of a triangle are distinct from the respectively opposite sides, they meet these sides in three collinear points.*

Proof. Let APX be a triangle whose sides PX, XA, AP meet the polars a, p, x of its vertices in points A_1, P_1, X_1, as in Figure 14.6*b*. The polar of $X_1 = x \cdot AP$ is, of course, $x_1 = X(a \cdot p)$. Define also the extra points $P' = a \cdot AP$, $X' = a \cdot AX$ and their polars $p' = A(a \cdot p)$, $x' = A(a \cdot x)$. By 14.54 and the polarity, we have

$$AP'PX_1 \barwedge P'AX_1P \barwedge p'ax_1p \barwedge AX'XP_1.$$

By 14.53, $AP'PX_1 \doublebarwedge AX'XP_1$. Since the center of this perspectivity is $P'X' \cdot PX = A_1$, the three points A_1, P_1, X_1 are collinear, as desired.

We are now ready for the construction (Figure 14.6*c*):

14.65 *In the polarity $(ABC)(Pp)$, the polar of a point X (not on AP, BP, or p) is the line X_1X_2 determined by*

* This is known as Chasles's theorem. The proof given in *The Real Projective Plane* [Coxeter **2**, p. 71] suffices for real geometry but not for the more general geometry which is developed here. Lemma 5.54 of that book is false in the finite geometry $PG(2, 3)$, which admits a quadrilateral whose three pairs of opposite vertices P_1P_2, P_3P_6, P_5P_9 are pairs of conjugate points in the polarity $P_i \to p_i$ although the four sides $P_1P_3P_9$, $P_2P_6P_9$, $P_2P_3P_5$, $P_1P_5P_6$ contain their respective poles P_0, P_7, P_8, P_{11}. (The remaining three of the thirteen points in this finite plane are the diagonal points of the quadrangle $P_0P_7P_8P_{11}$; their joins in pairs are the diagonals of the quadrilateral $p_0p_7p_8p_{11}$.) See also W. G. Brown, *Canadian Mathematical Bulletin*, **3** (1960), pp. 221–223.

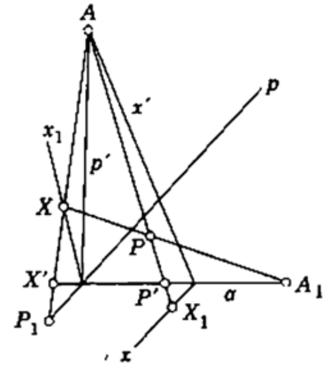


Figure 14.6b

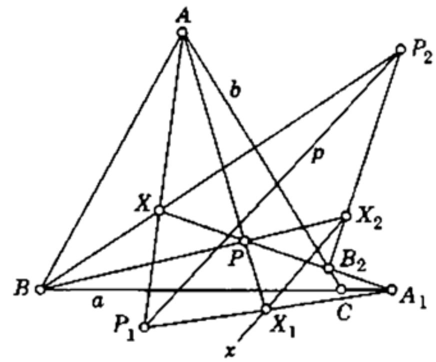


Figure 14.6c

$$A_1 = a \cdot PX, \quad P_1 = p \cdot AX, \quad X_1 = AP \cdot A_1P_1,$$
$$B_2 = b \cdot PX, \quad P_2 = p \cdot BX, \quad X_2 = BP \cdot B_2P_2.$$

Proof. By 14.64, the polars a, p, x meet the lines PX, AX, AP in three collinear points, the first two of which are A_1 and P_1. Hence x passes through $X_1 = AP \cdot A_1P_1$. Similarly x passes through $X_2 = BP \cdot B_2P_2$.

In terms of coordinates, a projective collineation is a linear homogeneous transformation

14.66 $$\mu x'_\alpha = \Sigma c_{\alpha\beta} x_\beta, \qquad \det(c_{\alpha\beta}) \neq 0,$$

where the summation is understood to be taken over the repeated index β (for each value of α). The nonvanishing of the determinant makes it possible to solve the equations for x_β in terms of x'_α so as to obtain the inverse collineation. By suitably adjusting the coefficients $c_{\alpha\beta}$, we can transform the particular quadrangle 14.23 into any given quadrangle [Coxeter **2,** p. 197].

Since the product of two correlations (e.g., a polarity and another correlation) is a collineation, any given projective correlation can be exhibited as the product of an arbitrary polarity and a suitable projective collineation. The most convenient polarity for this purpose is that in which the line

$$\Sigma X_\alpha x_\alpha = 0$$

is the polar of the point (X_1, X_2, X_3). Combining this with the general collineation 14.66, we obtain the correlation that transforms each point (y) into the line

14.661 $$\Sigma(\Sigma c_{\alpha\beta} y_\beta) x_\alpha = 0,$$

where again we must have $\det(c_{\alpha\beta}) \neq 0$. In fact, the correlation is associated with the *bilinear* equation

$$\Sigma\Sigma c_{\alpha\beta} x_\alpha y_\beta = 0$$

[cf. Coxeter **2,** p. 200].

The correlation is a polarity if it is the same as its *inverse*, whose equation, derived by interchanging (x) and (y), is

$$\Sigma\Sigma c_{\alpha\beta} y_\alpha x_\beta = 0, \quad \text{or} \quad \Sigma\Sigma c_{\beta\alpha} x_\alpha y_\beta = 0.$$

Thus a polarity occurs when $c_{\beta\alpha} = \lambda c_{\alpha\beta}$, where λ is the same for all α and β, so that $c_{\alpha\beta} = \lambda c_{\beta\alpha} = \lambda^2 c_{\alpha\beta}$, $\lambda^2 = 1$, $\lambda = \pm 1$. But we cannot have $\lambda = -1$, as this would make the determinant

$$\begin{vmatrix} 0 & c_{12} & -c_{31} \\ -c_{12} & 0 & c_{23} \\ c_{31} & -c_{23} & 0 \end{vmatrix} = 0.$$

Hence $\lambda = 1$, and $c_{\beta\alpha} = c_{\alpha\beta}$. In other words,

14.67 *A projective correlation is a polarity if and only if its matrix of coefficients is symmetric.*

Thus the general polarity is given by

14.68 $$\Sigma\Sigma c_{\alpha\beta} x_\alpha y_\beta = 0, \qquad c_{\beta\alpha} = c_{\alpha\beta}, \quad \det(c_{\alpha\beta}) \neq 0,$$

meaning that the polar of (y_1, y_2, y_3) is 14.661, or that 14.68 is the condition for points (x) and (y) to be conjugate. Setting $y_\beta = x_\beta$, we deduce the condition

$$\Sigma\Sigma c_{\alpha\beta} x_\alpha x_\beta = 0,$$

or $c_{11}x_1^2 + c_{22}x_2^2 + c_{33}x_3^2 + 2c_{23}x_2x_3 + 2c_{31}x_3x_1 + 2c_{12}x_1x_2 = 0$, for the point (x) to be self-conjugate. Hence

14.69 *If a polarity admits self-conjugate points, their locus is given by an equation of the second degree.*

EXERCISES

1. Given the center and axis of a perspective collineation, and one pair of corresponding points (collinear with the center), set up a construction for the transform X' of any point X [Coxeter **2**, p. 62].

2. Any two perspective triangles are related by a perspective collineation.

3. A collineation which leaves just the points of one line invariant is an elation.

4. An elation with axis o may be expressed as the product of two harmonic homologies having this same axis o [Coxeter **2**, p. 63].

5. In $PG(2, 3)$, the transformation $P_i \to P_{i+1}$ (with subscripts reduced modulo 13) is evidently a collineation of period 13. Is it a projective collineation? Consider also the transformation $P_i \to P_{3i}$.

6. What kind of collineation is

   (i) $x'_1 = x_1, \quad x'_2 = x_2, \quad x'_3 = cx_3$;

   (ii) $x'_1 = x_1 + c_1x_3, \quad x'_2 = x_2 + c_2x_3, \quad x'_3 = x_3$?

7. Use 14.64 to prove *Hesse's theorem*: If two pairs of opposite vertices of a complete quadrilateral are pairs of conjugate points (in a given polarity), then the third pair of opposite vertices is likewise a pair of conjugate points.

8. Give an analytic proof of Hesse's theorem. (*Hint*: Apply the condition 14.68 to the pairs of vertices

$$(0, 1, \pm 1), \quad (\pm 1, 0, 1), \quad (1, \pm 1, 0)$$

of the quadrilateral $x_1 \pm x_2 \pm x_3 = 0$.)

9. The bilinear equation

$$x_1y_1 + x_2y_2 + x_3y_3 = 0$$

is the condition for (x) and (y) to be conjugate in the polarity $(ABC)(Pp)$, where ABC is the triangle of reference, P is $(1, 1, 1)$, and p is $x_1 + x_2 + x_3 = 0$. Are there any self-conjugate points? Consider, in particular, the case when the coordinates are residues modulo 3.

14.7 THE CONIC

The three familiar curves which we call the "conic sections" have a long history. The reputed discoverer was Menaechmus, who flourished about 350 B.C. They attracted the attention of the best of the Greek geometers down to the time of Pappus of Alexandria. . . . A vivid new interest arose in the seventeenth century. . . . It seems certain that they will always hold a place in the mathematical curriculum.

J. L. Coolidge (1873-1954)

[Coolidge **1**, Preface]

In the projective plane there is only one kind of conic. The familiar distinction between the hyperbola, parabola, and ellipse belongs to affine geometry. To be precise, it depends on whether the line at infinity is a secant, a tangent, or a nonsecant [Coxeter **2**, p. 129].

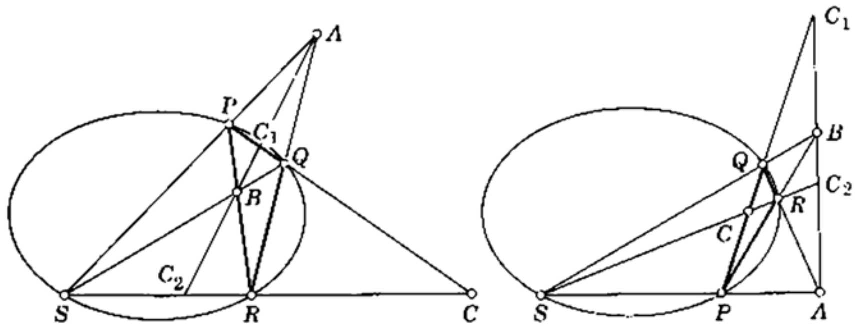


Figure 14.7a

A polarity is said to be *hyperbolic* or *elliptic* according as it does or does not admit a self-conjugate point. (In the former case it also admits a self-conjugate line: the polar of the point.) The self-conjugate point P, whose existence suffices to make a polarity hyperbolic, is by no means the only self-conjugate point: there is another on every line through P except its polar p. To prove this we use 14.63, which tells us that every such line contains an involution of conjugate points. By 14.57, this involution, having one invariant point P, has a second invariant point Q, which is, of course, another self-conjugate point of the polarity. Thus the presence of one self-conjugate point implies the presence of many (as many as the lines through a point; for example, infinitely many in real or complex geometry). Their

locus is a *conic*, and their polars are its *tangents*. This simple definition, due to von Staudt, exhibits the conic as a self-dual figure: the locus of self-conjugate points and also the envelope of self-conjugate lines.

The reader must bear in mind that there are only two kinds of polarity and that there is only one kind of conic. The terminology is perhaps not very well chosen: a hyperbolic polarity has many self-conjugate points, forming a conic; an elliptic polarity has no self-conjugate points at all, but still provides a polar for each point and a pole for each line; there is no such thing as a "parabolic polarity."

A tangent justifies its name by meeting the conic only at its pole, the *point of contact*. Any other line is called a *secant* or a *nonsecant* according as it meets the conic twice or not at all, that is, according as the involution of conjugate points on it is hyperbolic or elliptic. Any two conjugate points on a secant PQ, being paired in the involution $(PP)(QQ)$, are harmonic conjugates with respect to P and Q.

Let PQR be a triangle inscribed in a conic, as in Figure 14.7*a*. Any line c conjugate to PQ is the polar of some point C on PQ. Let RC meet the conic again in S. Then C is one of the three diagonal points of the inscribed quadrangle $PQRS$. The other two are

$$A = PS \cdot QR, \quad B = QS \cdot RP.$$

Their join meets the sides PQ and RS in points C_1 and C_2 such that $\mathsf{H}(PQ, CC_1)$ and $\mathsf{H}(RS, CC_2)$. Since C_1 and C_2 are conjugate to C, the line AB, which contains them, is c, the polar of C. Similarly BC is the polar of A. Therefore A and B are conjugate points. These conjugate points are the intersections of c with the sides QR and RP of the given triangle. Hence

SEYDEWITZ'S THEOREM. *If a triangle is inscribed in a conic, any line conjugate to one side meets the other two sides in conjugate points.*

From this we shall have no difficulty in deducing

STEINER'S THEOREM. *Let lines x and y join a variable point on a conic to two fixed points on the same conic; then $x \barwedge y$.*

Proof. The tangents p and q, at the fixed points P and Q, intersect in D, the pole of PQ. Let c be a fixed line through D (but not through P or Q), meeting x and y in B and A, as in Figure 14.7*b*. By Seydewitz's theorem,

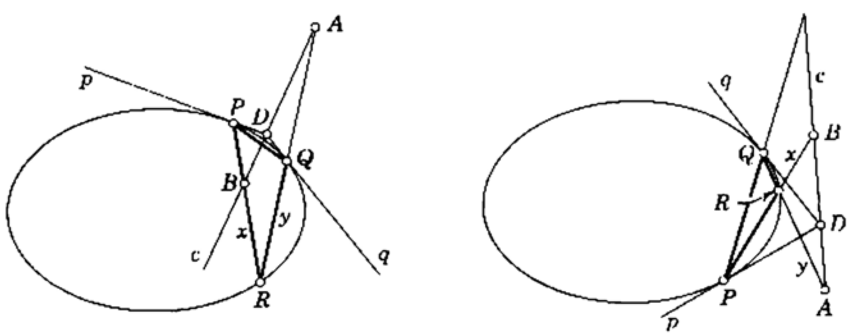


Figure 14.7b

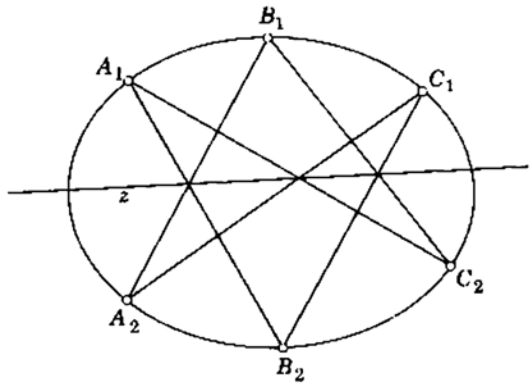


Figure 14.7c

BA is a pair of the involution of conjugate points on c. Hence, when the point $x \cdot y$ varies on the conic,

$$x \barwedge B \barwedge A \barwedge y.$$

The following construction for a conic through five given points, no three collinear, was discovered by Braikenridge and Maclaurin independently, about 1733 [Coxeter **2**, p. 91]. Let A_1, B_2, C_1, A_2, B_1 be the five points, as in Figure 14.7*c*; then the conic is the locus of the point

$$C_2 = A_1(z \cdot C_1A_2) \cdot B_1(z \cdot C_1B_2),$$

where z is a variable line through the point $A_1B_2 \cdot B_1A_2$. This is the converse of

PASCAL'S THEOREM. *If a hexagon $A_1B_2C_1A_2B_1C_2$ is inscribed in a conic, the points of intersection of pairs of opposite sides, namely,*

$$B_1C_2 \cdot B_2C_1, \quad C_1A_2 \cdot C_2A_1, \quad A_1B_2 \cdot A_2B_1,$$

are collinear.

Pascal discovered his famous theorem [Coxeter **2**, p. 103] when he was only sixteen years old. More than 150 years later, it was dualized (see Figure 14.7*d*):

BRIANCHON'S THEOREM. *If a hexagon is circumscribed about a conic, its three diagonals are concurrent.*

We saw, in § 8.4, that the familiar conics of Euclidean geometry have equations of the second degree in Cartesian coordinates. The same equations in affine coordinates remain valid in affine geometry, and yield homogeneous equations of the second degree in barycentric coordinates (§ 13.7) and in projective coordinates (§ 14.2). Thus 14.69 serves to reconcile von Staudt's definition of a conic with the classical definitions. In particular,

$$x_1x_3 = x_2^2$$

is a conic touching the lines $x_3 = 0$ and $x_1 = 0$ at the respective points

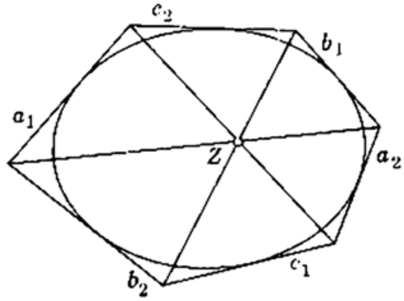


Figure 14.7d

(1, 0, 0) and (0, 0, 1). This conic can be parametrized in the form

$$x_1 : x_2 : x_3 = t^2 : t : 1,$$

which exhibits it as the locus of the point of intersection of corresponding members of the projectively related pencils of lines

$$x_1 = tx_2 \quad \text{and} \quad x_2 = tx_3.$$

If $\det(c_{\alpha\beta}) = 0$, the quadratic form $\Sigma\Sigma c_{\alpha\beta}x_\alpha x_\beta$ may be expressible as the product of two linear forms $\Sigma a_\alpha x_\alpha$ and $\Sigma b_\beta x_\beta$. Accordingly, a pair of lines is sometimes regarded as a degenerate conic. In this sense, Axiom 14.15 is a special case of Pascal's theorem.

EXERCISES

1. If a quadrangle is inscribed in a conic, its diagonal points form a self-polar triangle. The tangents at the vertices of the quadrangle form a circumscribed quadrilateral whose diagonals are the sides of the same triangle [Coxeter **2**, pp. 85, 86].

2. Referring to the projectivity $x \barwedge y$ of Steiner's theorem, investigate the special positions of x and y when A or B coincides with D.

3. If a projectivity between pencils of lines x and y through P and Q has the effect $xpd \barwedge ydq$, where d is PQ, the locus of the point $x \cdot y$ is a conic through P and Q whose tangents at these points are p and q. (This construction is often used to *define* a conic; see, e.g., Robinson [**1**, p. 38].)

4. Of the conics that touch two given lines at given points, those which meet a third line (not through either of the points) do so in pairs of an involution [Coxeter **2**, p. 90].

5. If two triangles are self-polar for a given polarity, their six vertices lie on a conic or on two lines [Coxeter **2**, p. 93].

6. If two triangles have six distinct vertices, all lying on a conic, they are self-polar for some polarity [Coxeter **2**, p. 94].

7. In $PG(2, 3)$ (Ex. 3 at the end of § 14.1), the polarity $P_i \to p_i$ or $(P_4P_{10}P_{12})(P_0p_0)$ determines a conic consisting of the four points P_0, P_7, P_8, P_{11} and the four lines p_0, p_7, p_8, p_{11}. (*Hint:* $P_0P_2P_8P_{12} \barwedge P_1P_7P_5P_4 \barwedge p_0p_2p_8p_{12}$.)

8. The equation $x_1^2 + x_2^2 - x_3^2 = 0$ represents a conic for which the triangle of reference is self-polar. Verify Pascal's theorem as applied to the inscribed hexagon

$$(0, 1, 1)\ (0, -1, 1)\ (1, 0, 1)\ (-1, 0, 1)\ (3, 4, 5)\ (4, 3, 5).$$

14.8 PROJECTIVE SPACE

Our Geometry is an abstract Geometry. The reasoning could be followed by a disembodied spirit who had no idea of a physical point; just as a man blind from birth could understand the Electromagnetic Theory of Light.

H. G. Forder [**1**, p. 43]

Axiom 14.12 had the effect of restricting the geometry to a single plane. If we remove this restriction, we must know exactly what we mean by a plane. First we define a *flat pencil* to be the set of lines joining a range of points (on a line) to another point. Then we define a *plane* to be the set of points on the lines of a flat pencil and the set of lines joining pairs of these points. Accordingly *we replace Axiom* 14.12 *by three new axioms*. The first (which may be regarded as a projective version of Pasch's axiom, 12.27) allows us to forget the role of a particular flat pencil in the definition of a plane. The second enables us to speak of more than one plane. The third (cf. 12.431) restricts the number of dimensions to three.

AXIOM 14.81 *If A, B, C, D are four distinct points such that AB meets CD, then AC meets BD.*

AXIOM 14.82 *There is at least one point not in the plane ABC.*

AXIOM 14.83 *Any two planes meet in a line.*

We now have a different principle of duality: points, lines and planes correspond to planes, lines and points (cf. § 10.5). Two intersecting lines, a and b, determine a point $a \cdot b$ and a plane ab; these are dual concepts. Two lines that do not intersect are said to be *skew*. The theory of collineations and correlations [Coxeter **3**, pp. 63–70] is analogous to the two-dimensional case, except that the number of self-conjugate points on a line is no longer restricted to 0, 1, or 2. In fact, instead of two kinds of polarity we now have four: one "elliptic," having no self-conjugate points, two "hyperbolic," whose self-conjugate points form a quadric (nonruled or ruled), and one, the *null polarity* (or "null system"), in which every point in space is self-conjugate!

The idea of defining a quadric as the locus of self-conjugate points in a three-dimensional polarity (of the second or third kind) is due to von Staudt. Another approach, using a two-dimensional polarity in an arbitrary plane ω, was devised by F. Seydewitz.* The quadric appears as the locus of the point

$$PA \cdot Qa,$$

where P and Q are fixed points (on the quadric) while A is a variable point on ω and a is its polar. This definition allows the quadric to degenerate to a cone or a pair of planes.

To sample the flavor of solid projective geometry, let us consider a few

* *Archiv für Mathematik und Physik*, **9** (1848), p. 158.

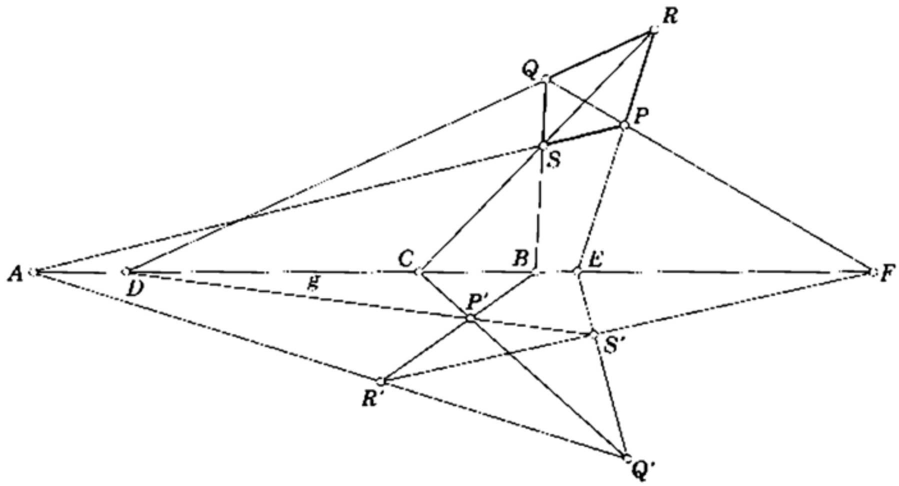


Figure 14.8a

typical theorems. Suppose a complete quadrangle $PQRS$ yields a quadrangular set $(AD)(BE)(CF)$ on a line g, as in Figure 14.8a. In another plane through g, let the sides of a triangle $P'Q'R'$ pass through A, B, C, and let DP' meet EQ' in S'. Theorem 14.42 tells us that S' lies on $R'F$. This remark yields two interesting configurations: one consisting of eight lines (Figure 14.8b), and the other of two mutually inscribed tetrahedra.

GALLUCCI'S THEOREM. *If three skew lines all meet three other skew lines, any transversal to the first set of three meets any transversal to the second set.*

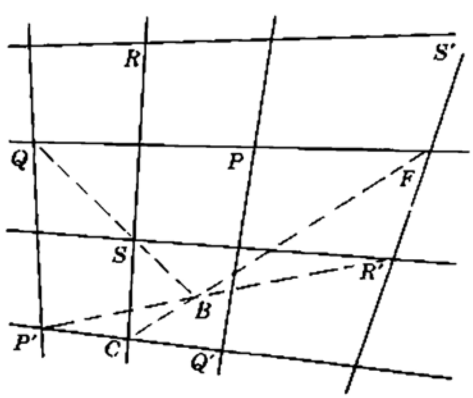
Proof. Let the two sets of lines be PQ', $P'Q$, RS; PQ, $P'Q'$, $R'S$. This notation agrees with Figure 14.8a, for, since PS and $Q'R'$ both pass through A, PQ' meets $R'S$, and since QS and $R'P'$ both pass through B, $P'Q$ meets $R'S$. The transversal from R to PQ' and $P'Q$ is

$$RPQ' \cdot RP'Q = REQ' \cdot RDP' = RS'.$$

The transversal from R' to PQ and $P'Q'$ is

$$R'PQ \cdot R'P'Q' = R'FQ \cdot R'FQ' = R'F.$$

Since S' lies on $R'F$, these transversals meet, as desired.



MÖBIUS'S THEOREM. *If the four vertices of one tetrahedron lie respectively in the four face planes of another, while three vertices of the second lie in three face planes of the first, then the remaining vertex of the second lies in the remaining face plane of the first.*

Proof. Let $PQRS'$ and $P'Q'R'S$ be the two tetrahedra, with

$$P, \quad Q, \quad R, \quad S', \quad P', \quad Q', \quad S$$

in the respective planes

$$Q'R'S,\ P'R'S,\ P'Q'S,\ P'Q'R',\ QRS',\ PRS',\ PQR,$$

as in Figure 14.8*a*. Since $R'S'$ passes through F, on PQ, the remaining vertex R' lies in the remaining plane PQS', as desired.

Changing the notation from

$$S, \quad P, \quad Q, \quad R, \quad P', \quad Q', \quad R', \quad S'$$

to

$$S, \quad S_{14}, \quad S_{24}, \quad S_{34}, \quad S_{23}, \quad S_{13}, \quad S_{12}, \quad S_{1234},$$

we deduce the first of a remarkable "chain" of theorems due to Homersham Cox:*

COX'S FIRST THEOREM. *Let $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ be four planes of general position through a point S. Let S_{ij} be an arbitrary point on the line $\sigma_i \cdot \sigma_j$. Let σ_{ijk} denote the plane $S_{ij}S_{ik}S_{jk}$. Then the four planes $\sigma_{234}, \sigma_{134}, \sigma_{124}, \sigma_{123}$ all pass through one point S_{1234}.*

Clearly, σ_1, σ_2, σ_3, σ_{123} are the face planes of the tetrahedron $P'Q'R'S$, while σ_{234}, σ_{134}, σ_{124}, σ_4 are those of the inscribed-circumscribed tetrahedron $PQRS'$. Let σ_5 be a fifth plane through S. Then S_{15}, S_{25}, S_{35}, S_{45} are four points in σ_5; σ_{ij5} is a plane through the line $S_{i5}S_{j5}$; and S_{ijk5} is the point $\sigma_{ij5} \cdot \sigma_{ik5} \cdot \sigma_{jk5}$. By the dual of Cox's first theorem, the four points S_{2345}, S_{1345}, S_{1245}, S_{1235} all lie in one plane. Interchanging the roles of σ_4 and σ_5, we see that S_{1234} lies in this same plane $S_{2345}S_{1345}S_{1245}$, which we naturally call σ_{12345}. Hence

COX'S SECOND THEOREM. *Let $\sigma_1, \ldots, \sigma_5$ be five planes of general position through S. Then the five points S_{2345}, S_{1345}, S_{1245}, S_{1235}, S_{1234} all lie in one plane σ_{12345}.*

Adding the extra digits 56 to all the subscripts in the first theorem, we deduce

COX'S THIRD THEOREM. *The six planes σ_{23456}, σ_{13456}, σ_{12456}, σ_{12356}, σ_{12346}, σ_{12345} all pass through one point S_{123456}.*

The pattern is now clear: we can continue indefinitely. "Cox's $(d-3)$rd

* *Quarterly Journal of Mathematics,* **25** (1891), p. 67. See also H. W. Richmond, *Journal of the London Mathematical Society,* **16** (1941), pp. 105–112, and Coxeter, *Bulletin of the American Mathematical Society,* **56** (1950), p. 446. When we describe four planes through a point as being "of general position," we mean that their six lines of intersection are all distinct.

theorem" provides a configuration of 2^{d-1} points and 2^{d-1} planes, with d of the planes through each point and d of the points in each plane.

Our next result would be difficult to obtain without using coordinates. Since the equation of the general quadric

$$c_{11}x_1^2 + \ldots + c_{44}x_4^2 + 2c_{12}x_1x_2 + \ldots + 2c_{34}x_3x_4 = 0$$

has $4 + 6 = 10$ terms, a unique quadric $\Sigma = 0$ can be drawn through nine points of general position; for, by substituting each of the nine given sets of x's in $\Sigma = 0$, we obtain nine linear equations to solve for the mutual ratios of the ten c's. Similarly, a "pencil" (or singly infinite system) of quadrics

$$\Sigma + \mu\Sigma' = 0$$

can be drawn through eight points of general position, and a "bundle" (or doubly infinite system) of quadrics

$$\Sigma + \mu\Sigma' + \nu\Sigma'' = 0$$

can be drawn through seven points of general position. But, by solving the simultaneous quadratic equations

$$\Sigma = 0, \quad \Sigma' = 0, \quad \Sigma'' = 0$$

for the mutual ratios of the four x's, we obtain eight points of intersection for these three quadrics. Naturally these eight points lie on every quadric of the bundle. Hence

Seven points of general position determine a unique eighth point, such that every quadric through the seven passes also through the eighth.

This idea of the eighth *associated* point provides an alternative proof for Cox's first theorem (and therefore also for the theorems of Möbius and Gallucci). Let S_{1234} be defined as the common point of the three planes $\sigma_{234}, \sigma_{134}, \sigma_{124}$. (The theorem states that S_{1234} lies also on σ_{123}.) Since the plane pairs $\sigma_1\sigma_{234}, \sigma_2\sigma_{134}, \sigma_3\sigma_{124}$ form three degenerate quadrics through the eight points

$$S, S_{14}, S_{24}, S_{34}, S_{23}, S_{13}, S_{12}, S_{1234},$$

these are eight associated points. The first seven belong also to the plane pair $\sigma_4\sigma_{123}$. Since S_{1234} does not lie in σ_4, it must lie in σ_{123}, as desired.

The locus of lines meeting three given skew lines is called a *regulus*. Gallucci's theorem shows that the lines meeting three generators of the regulus (including the original three lines) form another "associated" regulus, such that every generator of either regulus meets every generator of the other. The two reguli are the two systems of generators of a *ruled quadric*.

Let a_1, b_1, c_1, d_1 be four generators of the first regulus, and a_2, b_2, c_2, d_2 four generators of the second, as in Figure 14.8*c*. The three lines

$$a_3 = b_1c_2 \cdot b_2c_1, \quad b_3 = c_1a_2 \cdot c_2a_1, \quad c_3 = a_1b_2 \cdot a_2b_1$$

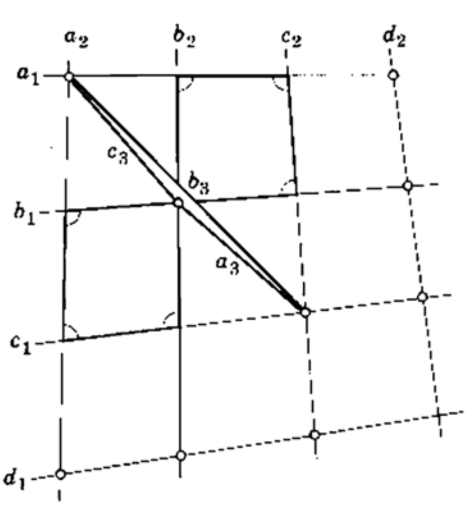


Figure 14.8c

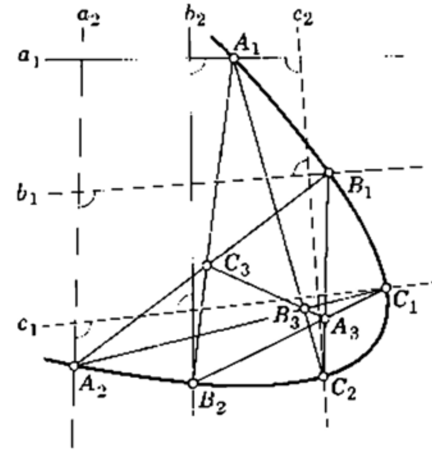


Figure 14.8d

evidently form a triangle whose vertices are $a_1 \cdot a_2$, $b_1 \cdot b_2$, $c_1 \cdot c_2$. G. P. Dandelin, in 1824, coined the name *hexagramme mystique* for the skew hexagon $a_1b_2c_1a_2b_1c_2$. Taking the section of its sides by a plane δ of general position, he obtained a plane hexagon $A_1B_2C_1A_2B_1C_2$ whose sides A_1B_2, B_2C_1, . . . lie in the planes a_1b_2, b_2c_1, . . . (Figure 14.8d). The points of intersection of pairs of opposite sides, namely,

$$A_3 = B_1C_2 \cdot B_2C_1, \quad B_3 = C_1A_2 \cdot C_2A_1, \quad C_3 = A_1B_2 \cdot A_2B_1,$$

each lying in both the planes $a_3b_3c_3$ and δ, are collinear. By allowing c_2 to vary while the remaining sides of the skew hexagon remain fixed, we see from the Braikenridge-Maclaurin construction (which is the converse of Pascal's theorem, Figure 14.7c), that

The section of a ruled quadric, by a plane of general position, is a conic.

If δ, instead of being a plane of general position, is the plane d_1d_2, the vertices of the hexagon $A_1B_2C_1A_2B_1C_2$ line alternately on d_2 and d_1, as in Axiom 14.15. Thus Pappus's theorem may be regarded as a "degenerate" case of Pascal's theorem. In fact, instead of assuming Pappus's theorem and deducing Gallucci's theorem, we could have taken the latter as an axiom and deduced the former. Bachmann [**1**, p. 254] gives a particularly fine figure to illustrate this deduction.

EXERCISES

1. If a and b are two skew lines and R is a point not on either of them, $Ra \cdot Rb$ is the *only* transversal from R to the two lines.

2. Any plane through a generator of a ruled quadric contains another generator. (Such a plane is a *tangent* plane.) Any other plane section of the ruled quadric is a conic.

3. If two tetrahedra are trebly perspective they are quadruply perspective (cf. § 14.3). More precisely, if $A_1A_2A_3A_4$ is perspective with each of $B_2B_1B_4B_3$, $B_3B_4B_1B_2$,

$B_4B_3B_2B_1$, it is also perspective with $B_1B_2B_3B_4$. (*Hint*: Since A_iB_j meets A_jB_i, A_iB_i must meet A_jB_j.)

4. The four centers of perspective that were implied in Ex. 3 form a third tetrahedron which is perspective with either of the first two from each vertex of the remaining one.

5. In the finite space $PG(3, 3)$, which has 4 points on each line, there are altogether 40 points, 40 planes, and how many lines?

14.9 EUCLIDEAN SPACE

> *The set of lines drawn from the artist's eye to the various points of the object . . . constitute the projection of the object and are called the Euclidean cone. Then the section of this cone made by the canvas is the desired drawing. . . . Parallel lines in the object converge in the picture to the point where the canvas is pierced by the line from the eye parallel to the given lines.*
>
> S. H. Gould [**1**, p. 299]

The elementary approach to affine space is to regard it as Euclidean space without a metric; the elementary approach to projective space is to regard it as affine space plus the plane at infinity and then to ignore the special role of that plane. It is equally effective to begin with projective space and derive affine space by specializing any one plane, calling it the plane at infinity. (This is still, of course, a projective plane.) Each affine concept has its projective definition: for example, the midpoint of AB is the harmonic conjugate, with respect to A and B, of the point at infinity on AB [Coxeter **2**, p. 119]. We then derive Euclidean space by specializing one elliptic polarity in the plane at infinity, calling it the *absolute polarity*. Two lines are orthogonal if their points at infinity are conjugate in the absolute polarity; a line and a plane are orthogonal if the point at infinity on the line is the pole of the line at infinity in the plane. A sphere is the locus of the point of intersection of a line through one fixed point and the perpendicular plane through another; thus it is a special quadric according to Seydewitz's definition. Two segments with a common end are congruent if they are radii of the same sphere [Coxeter **2**, p. 146].

When we use projective coordinates (x_1, x_2, x_3, x_4), referred to an arbitrary tetrahedron

$$(1, 0, 0, 0)\ (0, 1, 0, 0)\ (0, 0, 1, 0)\ (0, 0, 0, 1),$$

it is convenient to take the plane at infinity to be $x_4 = 0$. Any other equation becomes an equation in affine coordinates x_1, x_2, x_3 by the simple device of setting $x_4 = 1$. In affine terms, the tetrahedron of reference for the projective coordinates is formed by the origin and the points at infinity on the three axes. Finally, we pass from affine space to Euclidean space by

declaring that two points (x_1, x_2, x_3) and (y_1, y_2, y_3) are in perpendicular directions from the origin if they satisfy the bilinear equation

$$x_1y_1 + x_2y_2 + x_3y_3 = 0,$$

that is, if the points at infinity

$$(x_1, x_2, x_3, 0) \quad \text{and} \quad (y_1, y_2, y_3, 0)$$

are conjugate in the absolute polarity.

All the theorems that we proved in § 14.8 remain valid in Euclidean space. An interesting variant of Cox's chain of theorems can be obtained by means of the following specialization. Instead of an *arbitrary* point on the line $\sigma_i \cdot \sigma_j$, we take S_{ij} to be the second intersection of this line with a fixed sphere through S. Since the sphere is a quadric through the first seven of the eight associated points

$$S,\ S_{14},\ S_{24},\ S_{34},\ S_{23},\ S_{13},\ S_{12},\ S_{1234},$$

it passes through S_{1234} too, and similarly through S_{1235} and the rest of the 2^{d-1} points. The 2^{d-1} planes meet the sphere in 2^{d-1} circles, which remain circles when we make an arbitrary stereographic projection, as in § 6.9. We thus obtain Clifford's chain of theorems* in the inversive (or Euclidean) plane.

CLIFFORD'S FIRST THEOREM. *Let $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ be four circles of general position through a point S. Let S_{ij} be the second intersection of the circles σ_i and σ_j. Let σ_{ijk} denote the circle $S_{ij}S_{ik}S_{jk}$. Then the four circles $\sigma_{234}, \sigma_{134}, \sigma_{124}, \sigma_{123}$ all pass through one point S_{1234}.*

CLIFFORD'S SECOND THEOREM. *Let σ_5 be a fifth circle through S. Then the five points $S_{2345}, S_{1345}, S_{1245}, S_{1235}, S_{1234}$ all lie on one circle σ_{12345}.*

CLIFFORD'S THIRD THEOREM. *The six circles $\sigma_{23456}, \sigma_{13456}, \sigma_{12456}, \sigma_{12356}, \sigma_{12346}, \sigma_{12345}$ all pass through one point S_{123456}.*

And so on!

EXERCISES

1. Why is the absolute polarity elliptic?
2. Draw a careful figure for Clifford's first theorem.
3. The circumcircles of the four triangles formed by four general lines all pass through one point (cf. Ex. 2 at the end of § 5.5).
4. The circumcenters of the four triangles of Ex. 3 all lie on a circle which passes also through the point of concurrence of the four circumcircles [Forder **3**, pp. 16–22; Baker **1**, p. 328].

* W. K. Clifford, *Mathematical Papers* (London, 1882), p. 51. Apparently Clifford did not state these theorems in their full generality. Instead of circles through S he took $\sigma_1, \sigma_2, \ldots$ to be straight lines. In other words, he took S to be the point at infinity of the inversive plane. Thus his special form of the theorems could have been derived from the configuration of circles on the sphere by taking the center of the stereographic projection to be the point S on the sphere [Baker **1**, p. 133].

15

Absolute geometry

In the present chapter we shall re-examine the material of some of the earlier chapters in the light of the axiomatic approach outlined in Chapter 12, regarding classical geometry as ordered geometry enriched with the axioms of congruence 15.11–15.15, the last of which is a restatement of 1.26. Except in §§ 15.6 and 15.8, we shall work in the domain of *absolute* geometry, that is, we shall take care not to assume any form of Euclid's fifth postulate. Accordingly, our results will be valid not only in Euclidean geometry but also in the non-Euclidean geometry of Gauss, Lobachevsky, and Bolyai.

In § 15.4 we shall give a simple account of the complete enumeration of finite groups of isometries. According to Weyl [**1**, p. 79], "This is the modern equivalent to the tabulation of the regular polyhedra by the Greeks." The relevance of these kinematical results to crystallography makes it natural, in § 15.6, to reintroduce the full machinery of Euclidean geometry. But in § 15.7 we shall return to absolute geometry for a discussion of finite groups generated by reflections. Many of the methods used remain valid also in spherical geometry.

15.1 CONGRUENCE

Every teacher certainly should know something of non-Euclidean geometry. . . . It forms one of the few parts of mathematics which . . . is talked about in wide circles, so that any teacher may be asked about it at any moment.

F. Klein [**2,** p. 135]

To give a rigorous approach to absolute geometry, we begin with ordered geometry (Chapter 12) and introduce *congruence* as a third primitive concept: an undefined equivalence relation among point pairs (or segments, or intervals). We use the notation $AB \equiv CD$ to mean "AB is congruent to CD." The following axioms are those of Pasch with some refinements due to Hilbert and R. L. Moore [see Kerékjártó **1,** pp. 90–101].

Axioms of Congruence

15.11 *If A and B are distinct points, then on any ray going out from C there is just one point D such that $AB \equiv CD$.*

15.12 *If $AB \equiv CD$ and $CD \equiv EF$, then $AB \equiv EF$.*

15.13 $AB \equiv BA$.

15.14 *If $[ABC]$ and $[A'B'C']$ and $AB \equiv A'B'$ and $BC \equiv B'C'$, then $AC \equiv A'C'$.*

15.15 *If ABC and $A'B'C'$ are two triangles with $BC \equiv B'C'$, $CA \equiv C'A'$, $AB \equiv A'B'$, while D and D' are two further points such that $[BCD]$ and $[B'C'D']$ and $BD \equiv B'D'$, then $AD \equiv A'D'$.*

By two applications of 15.13, we have $AB \equiv AB$; that is, congruence is *reflexive*. From 15.11 and 15.12 we easily deduce that the relation $AB \equiv CD$ implies $CD \equiv AB$; that is, congruence is *symmetric*. Axiom 15.12 itself says that congruence is *transitive*. Hence congruence is an *equivalence* relation. This result, along with the *additive* property of 15.14, provides the basis for a theory of *length* [Forder **1**, p. 95]. Axiom 15.15 enables us to extend the relation of congruence from point pairs or segments to *angles* [Forder **1**, p. 132].

We follow Euclid in defining a *right angle* to be an angle that is congruent to its supplement; and we agree to measure angles on such a scale that the magnitude of a right angle is $\frac{1}{2}\pi$.

The statement $AB \equiv CD$ for segments is clearly equivalent to the statement $AB = CD$ for lengths, so no confusion arises from using the same symbol AB for a segment and its length. A similar remark applies to angles.

The *circle* with center O and radius r is defined as the locus of a variable point P such that $OP = r$. A point Q such that $OQ > r$ is said to be *outside* the circle. Points neither on nor outside the circle are said to be *inside*. It can be proved [Forder **1**, p. 131] that if a circle with center A has a point inside and a point outside a circle with center C, then the two circles meet in just one point on each side of the line AC. Euclid's first four postulates may now be treated as theorems, and we can prove all his propositions as far as I.26; also I.27 and 28 with the word "parallel" replaced by "nonintersecting." We can define reflection as in § 1.3, and derive its simple consequences such as *pons asinorum* (Euclid I.5) and the symmetry of a circle about its diameters (III.3; see § 1.5). But we must be careful to avoid any appeal to our usual idea about the sum of the angles of a triangle; for example, we can no longer assert that angles in the same segment of a circle are equal (Euclid III.21). Lacking such theorems as VI.2–4, which depend on the affine properties of parallelism, we have to look for some quite different way to prove the concurrence of the medians of a triangle.* On the other hand, the concurrence of the *altitudes* (of an acute-angled triangle) arises as a byproduct of Fagnano's problem, which can still be treated as in § 1.8. (Fermat's problem would require a different treatment because we can no longer assume the angles of an equilateral triangle to be $\pi/3$.)

* Bachmann **1**, pp. 74–75.

EXERCISES

1. Complete the proof that congruence is symmetric: if $AB \equiv CD$ then $CD \equiv AB$.

2. How much of § 1.5 remains valid in absolute geometry? [Kerékjártó **1**, pp. 161–163.] (See especially Exercises 1 and 3.)

3. For any simple quadrangle inscribed in a circle, the sum of two opposite angles is equal to the sum of the remaining two angles [Sommerville **1**, p. 84].

15.2 PARALLELISM

I have resolved to publish a work on the theory of parallels as soon as I have put the material in order. . . . The goal is not yet reached, but I have made such wonderful discoveries that I have been almost overwhelmed by them. . . . I have created a new universe from nothing.

Janos Bolyai (1802-1860)

(From a letter to his father in 1823)

Following Gauss, Bolyai, and Lobachevsky, we say that two lines are *parallel* if they "almost meet." For the precise meaning of this phrase, see § 12.6. (We use the notation p_1 for one of the two rays into which the line p is decomposed by a point that lies on it.)

The idea of the *incenter* (§ 1.5) may be extended from a triangle to the figure formed by two parallel lines and a transversal, enabling us to prove that parallelism is symmetric:

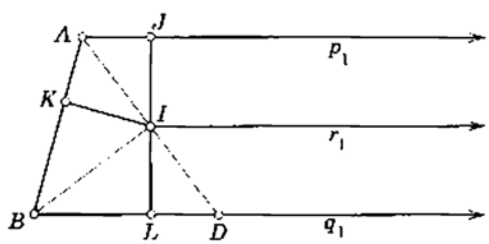


Figure 15.2a

15.21 *If p_1 is parallel to q_1, then q_1 is parallel to p_1.*

Proof. [Sommerville **1**, p. 32]. If p_1, through A, is parallel to q_1, through B, as in Figure 15.2a, the internal bisector AD of the angle at A completes a triangle ABD. Let the internal bisector of B meet AD in I. Draw perpendiculars IJ, IK, IL, to p_1, AB, q_1. Reflecting in IA and IB, we see that $IJ = IK = IL$. Let r_1 be the internal bisector of $\angle LIJ$. Reflection in the line r interchanges J and L, and therefore interchanges p and q. Since p is parallel to q, it follows that q is parallel to p in the same sense, that is, q_1 is parallel

to p_1. (In the terminology of Gauss, J and L are *corresponding* points on the two parallel rays.)

We can now use the methods of ordered geometry to prove that parallelism is transitive:

15.22 *If p_1 is parallel to q_1, and q_1 is parallel to r_1, then p_1 is parallel to r_1.*

Proof [Gauss **1**, vol. 8, pp. 205–206]. We have to show that, if p_1 and r_1 are both parallel to q_1, they are parallel to each other. We see at once that p_1 and r_1 cannot meet; for if they did, we would have two intersecting lines p and r both parallel to q in the given sense. By Theorem 12.64, we may assume that p_1, q_1, r_1 begin from three collinear points A, B, C. For the rest of the proof we distinguish the case in which B lies between A and C from the case in which it does not.

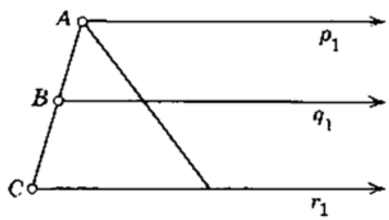


Figure 15.2b

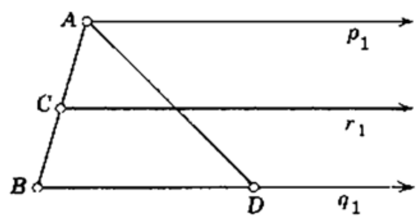


Figure 15.2c

If $[ABC]$, as in Figure 15.2*b*, any ray from A within the angle between AC and p_1 meets q_1 (since p_1 is parallel to q_1) and then meets r_1 (since q_1 is parallel to r_1). Therefore p_1 is parallel to r_1.

If B is not between A and C, suppose for definiteness that $[ACB]$, as in Figure 15.2*c*. Any ray from A within the angle between AC and p_1 meets q_1, say in D. Since r separates A from D, it meets the segment AD. Therefore p_1 is parallel to r_1.

In this second part of the proof we have not used the parallelism of q_1 and r_1. In fact,

15.23 *If a ray r_1 lies between two parallel rays, it is parallel to both.*

Having proved that parallelism is an equivalence relation, we consider the set of lines parallel to a given ray. We naturally call this a *pencil of parallels*, since it contains a unique line through any given point [Coxeter **2**, p. 5]. Pursuing its analogy with an ordinary pencil (consisting of all the lines through a point), we may also call it a *point at infinity* or, following Hilbert, an *end*. Instead of saying that two rays (or lines) are parallel, or that they belong to a certain pencil of parallels M, we say that they have M for a common end. In the same spirit, the ray through A that belongs to the given pencil of parallels is denoted by AM, as if it were a segment; the same symbol AM can also be used for the whole line.

Let AM, BM be parallel rays, and ε an arbitrarily small angle. Within the angle BAM (Figure 15.2*d*), take a ray from A making with AM an angle less than ε. This ray cuts BM in some point C. On CM (which is C/B), take D so that $CD = CA$. The isosceles triangle CAD yields

$$\angle ADC = \angle CAD < \angle CAM < \varepsilon.$$

Hence, when BD tends to infinity, so that AD tends to the position AM, $\angle ADB$ tends to zero.

This conclusion motivates the following assertion of Bolyai [**1**, p. 207]:

15.24 *When two parallel lines are regarded as meeting at infinity, the angle of intersection must be considered as being equal to zero.*

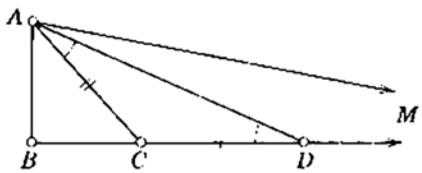


Figure 15.2d

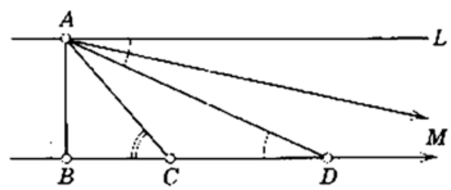


Figure 15.2e

When AM and BM are parallel rays, we call the figure ABM an *asymptotic triangle*. Such triangles behave much like finite triangles. In particular, two of them are congruent if they agree in the finite side and one angle [Carslaw **1**, p. 49]:

15.25 *If two asymptotic triangles ABM, $A'B'M'$ have $AB = A'B'$ and $A = A'$, then also $B = B'$.*

It is a consequence of Axiom 15.11 that, if two lines have a common perpendicular, they do not intersect. The following theorem provides a kind of converse for this statement.

15.26 *If two lines are neither intersecting nor parallel, they have a common perpendicular.*

Proof. From A on the first line AL, draw AB perpendicular to the second line BM, as in Figure 15.2*e*. If AB is perpendicular to AL there is no more to be said. If not, suppose L is on that side of AB for which $\angle BAL$ is acute. Since the two lines are neither intersecting nor parallel, there is a smaller angle BAM such that AM is parallel to BM. If $[BCD]$ on BM, we can apply Euclid I.16 to the triangle ACD, with the conclusion that the internal angle at D is less than the external angle at C. Hence, when BD increases from 0 to ∞, so that $\angle DAL$ decreases from $\angle BAL$ to $\angle MAL$, $\angle ADB$ decreases from a right angle to zero. At the beginning of this process we have

$$\angle DAL < \angle ADB$$

(since $\angle BAL$ is acute); but at the end the inequality is reversed (since

$\angle MAL$ is positive). Hence there must be some intermediate position for which

$$\angle DAL = \angle ADB.$$

(To be precise, we can apply Dedekind's axiom 12.51 to the points on BM satisfying the two opposite inequalities.) For such a point D (Figure 15.2*f*) we obtain two triangles OAE, ODF by drawing EF perpendicular to BD through O, the midpoint of AD. Since these triangles are congruent, EF is perpendicular not only to BD but also to AL.

Nonintersecting lines that are not parallel are said to be *ultraparallel* (or "hyperparallel"). We are not asserting the existence of such lines, but merely showing how they must behave if they do exist.

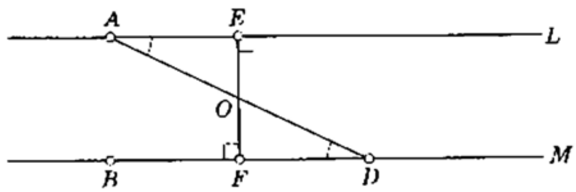


Figure 15.2f

EXERCISES

1. Prove 15.25 without referring to Carslaw **1**.

2. Give a complete proof that, if two lines have a common perpendicular, they do not intersect.

3. Example 4 on p. 16 remains valid when A is an end so that the triangle is asymptotic.

15.3 ISOMETRY

Beside the actual universe I can set in imagination other universes in which the laws are different.

J. L. Synge [**2**, p. 21]

The whole theory of finite groups of isometries (§§ 2.3–3.1) belongs to absolute geometry, because it is concerned with isometries having at least one invariant point. The first departure from our previous treatment (§ 3.2) is in the discussion of isometries without invariant points. We must now distinguish between a *translation*, which is the product of half-turns about two distinct points, and a *parallel displacement*, which is the product of reflections in two parallel lines.

The product of half-turns about two distinct points O, O' is a translation along a given line (called the *axis* of the translation) in a given sense through a given distance, namely, along OO' in the sense of the ray O'/O through the distance $2OO'$. Since a translation is determined by its axis and directed

distance, the product of half-turns about O, O' is the same as the product of half-turns about Q, Q', provided the directed segment QQ' is congruent to OO' *on the same line* (Figure 3.2*a*). If P is on this line, the distance PP^{T} is just twice OO'. (If not, it may be greater!)

By the argument used in proving 3.21, the product of two translations with the same axis, or with intersecting axes, is a translation. (It is only in the former case that we can be sure of commutativity.) More precisely, we have

15.31 (Donkin's theorem*) *The product of three translations along the directed sides of a triangle, through twice the lengths of these sides, is the identity.*

We shall see later that the product of two translations with nonintersecting axes may be a rotation.

By the argument used in proving 3.22, if two lines have a common perpendicular, the product of reflections in them is a translation along this common perpendicular through twice the distance between them. (Such lines may be either parallel or ultraparallel according to the nature of the geometry.)

Again, as in 3.13, every isometry is the product of at most three reflections. If the isometry is direct, the number of reflections is even, namely 2. It follows from 15.26 that

15.32 *Every direct isometry (of the plane) with no invariant point is either a parallel displacement or a translation.*

It is remarkable that absolute geometry includes the whole theory of glide reflection. The only changes needed in the previous treatment (§ 3.3) are where the word "parallel" was used. (In Figure 3.3*b* we must define m, m' as being perpendicular to OO'; they are not necessarily parallel to each other.) As an immediate application of these ideas we have Hjelmslev's theorem, which is one of the best instances of a genuinely surprising result belonging to absolute geometry. The treatment in § 3.6 remains valid without changing a single word!

Likewise, the one-dimensional groups of § 3.7 belong to absolute geometry, the only change being that again the mirrors m, m' (Figure 3.7*b*) should not be said to be "parallel" but both perpendicular to the same (horizontal) line. On the other hand, the whole theory of lattices (Chapter 4) and of similarity (Chapter 5) must be abandoned.

The extension of absolute geometry from two dimensions to three presents no difficulty. In particular, much of the Euclidean theory of isometry (§ 7.1) remains valid in absolute space. It is still true that every direct isometry is the product of two half-turns, and that every opposite isometry with

* W. F. Donkin, On the geometrical theory of rotation, *Philosophical Magazine* (4), **1**, (1851), 187–192. Lamb [**1**, p. 6] used half-turns about the vertices A, B, C of the given triangle to construct three new triangles which, he said, "are therefore directly equal to one another, and 'symmetrically' equal to ABC." This was a mistake: all four triangles are directly congruent!

an invariant point is a rotatory inversion (possibly reducing to a reflection or to a central inversion). Moreover, the classical enumeration of the five Platonic solids (§§ 10.1–10.3) is part of absolute geometry. The few necessary changes are easily supplied; for example, the term *rectangle* must be interpreted as meaning a quadrangle whose angles are all equal (though not necessarily right angles), and a *square* is the special case when also the sides are equal.

EXERCISES

1. If l is a line outside the plane of a triangle ABC, what can be said about the three lines in which this plane meets the three planes Al, Bl, Cl? (If two of the three lines intersect, or are parallel, or have a common perpendicular, the same can be said of all three. This property of three lines m_1, m_2, m_3 is equivalent to $\mathbf{R}_1\mathbf{R}_2\mathbf{R}_3 = \mathbf{R}_3\mathbf{R}_2\mathbf{R}_1$ in the notation of § 3.4.)

2. The product of reflections in the lines p and r of Figure 15.2a is a parallel displacement which transforms J into L.

15.4 FINITE GROUPS OF ROTATIONS

These groups, in particular the last three, are an immensely attractive subject for geometric investigation.

H. Weyl [**1**, p. 79]

One of the simplest kinds of transformation is a *permutation* (or rearrangement) of a finite number of named objects. For instance, one way to permute the six letters a, b, c, d, e, f is to transpose (or interchange) a and b, to change c into d, d into e, e into c, and to leave f unaltered. This permutation is denoted by $(a\ b)(c\ d\ e)$. The two "independent" parts, $(a\ b)$ and $(c\ d\ e)$, are called *cycles* of periods 2 and 3. A permutation that consists of just one cycle is said to be *cyclic*. Clearly, the cyclic group C_n may be represented by the powers of the generating permutation $(a_1a_2 \dots a_n)$; for instance, the four elements of C_4 are

$$1, \quad (a\ b\ c\ d), \quad (a\ c)(b\ d), \quad (a\ d\ c\ b).$$

A cyclic permutation of period 2, such as $(a\ b)$, is called a *transposition*. Since

$$(a_1a_2 \dots a_n) = (a_1a_n)(a_2a_n) \dots,$$

any permutation may be expressed as a product of transpositions. A permutation is said to be *even* or *odd* according to the parity of the number of cycles of even period; for instance, $(a\ c)(b\ d)$ is even, but $(a\ b)(c\ d\ e)$ is odd. The identity, 1, has no cycles at all, and is accordingly classified as an even permutation. It is easily proved [see Coxeter **1**, pp. 40–41] that every product of transpositions is even or odd according to the parity of the number of transpositions. It follows that the multiplication of even and odd per-

mutations behaves like the *addition* of even and odd numbers; for example, the product of two odd permutations is even.

It follows also that every group of permutations either consists entirely of even permutations or contains equal numbers of even and odd permutations. The group of all permutations of n objects is called the *symmetric* group of order $n!$ (or of *degree* n) and is denoted by S_n. The subgroup consisting of all the even permutations is called the *alternating* group of order $\frac{1}{2}n!$ (or of degree n) and is denoted by A_n. In particular, S_2 is the same group as C_2, and A_3 the same as C_3, so we write

$$S_2 \cong C_2, \qquad A_3 \cong C_3.$$

More interestingly, $S_3 \cong D_3$ (see Figure 2.7*a*). For, the six elements of the dihedral group D_3, being symmetry operations of an equilateral triangle, may be regarded as permutations of the three sides of the triangle. The even permutations

$$1, \quad (a\ b\ c), \quad (a\ c\ b)$$

(which form the subgroup $A_3 \cong C_3$) are rotations, whereas the odd permutations

$$(b\ c), \quad (c\ a), \quad (a\ b)$$

are reflections in the three medians. If we regard the triangle as lying in three-dimensional (absolute) space, the rotations are about an axis through the center of the triangle, perpendicular to its plane. The reflections may then be interpreted in two alternative ways, yielding two groups which are geometrically distinct but abstractly identical or *isomorphic*: we may either reflect in three planes through the axis or rotate through half-turns about the medians themselves. In the latter representation, all the six elements of D_3 appear as rotations. We may describe this as the group of direct symmetry operations of a triangular prism. More generally, the $2n$ direct symmetry operations of an n-gonal prism form the dihedral group D_n, whereas of course the n direct symmetry operations of an n-gonal pyramid form the cyclic group C_n. The rotations of C_n all have the same axis, and D_n is derived from C_n by adding half-turns about n lines symmetrically disposed in a plane perpendicular to that axis.

We have thus found two infinite families of finite groups of rotations. Other such groups are the groups of direct symmetry operations of the five Platonic solids $\{p, q\}$. These are only three groups, not five, because any rotation that takes $\{p, q\}$ into itself also takes the reciprocal $\{q, p\}$ into itself: the octahedron has the same group of rotations as the cube, and the icosahedron the same as the dodecahedron.

The regular tetrahedron $\{3, 3\}$ is evidently symmetrical by reflection in the plane that joins any edge to the midpoint of the opposite edge. As a permutation of the four faces a, b, c, d (Figure 15.4*a*), this reflection is just a transposition. Thus the complete symmetry group of the tetrahedron,

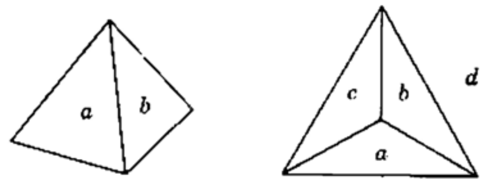


Figure 15.4a

being generated by such reflections, is isomorphic to the symmetric group S_4, which is generated by transpositions; and the rotation group, being generated by products of pairs of reflections, is isomorphic to the alternating group A_4, which is generated by products of pairs of transpositions. The 12 rotations may be counted as follows. The perpendicular from a vertex to the opposite face is the axis of a *trigonal* rotation (i.e., a rotation of period 3); the 4 vertices yield 8 such rotations. The line joining the midpoint of two opposite edges is the axis of a half-turn (or *digonal* rotation); the 3 pairs of opposite edges yield 3 such half-turns. Including the identity, we thus have $8 + 3 + 1 = 12$ rotations. As permutations, the 8 trigonal rotations are

$$(b\ c\ d),\quad (b\ d\ c),\quad (a\ c\ d),\quad (a\ d\ c),\quad (a\ b\ d),\quad (a\ d\ b),\quad (a\ b\ c),\quad (a\ c\ b)$$

and the 3 half-turns are

$$(b\ c)(a\ d),\quad (c\ a)(b\ d),\quad (a\ b)(c\ d).$$

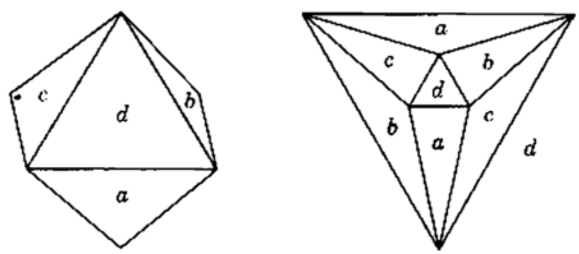


Figure 15.4b

The octahedron $\{3, 4\}$ can be derived from the tetrahedron by *truncation*: its eight faces consist of the four vertex figures of the tetrahedron and truncated versions of the four faces. Every symmetry operation of the tetrahedron is retained as a symmetry operation of the octahedron, but the octahedron also has symmetry operations that interchange the two sets of four faces. For instance, the line joining two opposite vertices is the axis of a *tetragonal* rotation (of period 4), and the line joining the midpoints of two opposite edges is the axis of a half-turn. When the four pairs of opposite faces are marked a, b, c, d, as in Figure 15.4b, such a half-turn appears as a transposition, which is one of the permutations that belong to S_4 but not

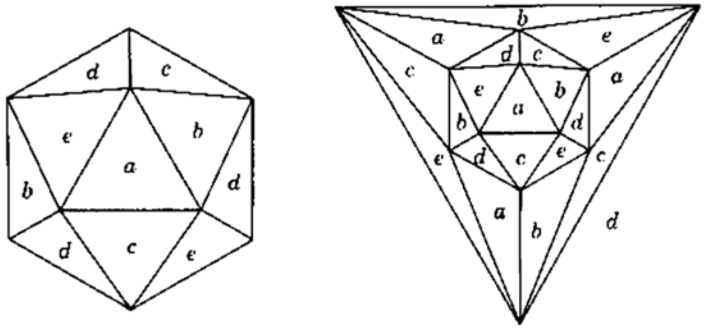


Figure 15.4c

to A_4. It follows that the rotation group of the octahedron (or of the cube) is isomorphic to the symmetric group S_4.

In Figure 15.4*c*, the twenty faces of the icosahedron $\{3, 5\}$ have been marked a, b, c, d, e in sets of four, in such a way that two faces marked alike have nothing in common, not even a vertex. In fact, the four a's (for instance) lie in the planes of the faces of a regular tetrahedron, and the respectively opposite faces (marked b, c, d, e) form the reciprocal tetrahedron. The twelve rotations of either tetrahedron into itself (represented by the even permutations of b, c, d, e) are also symmetry operations of the whole icosahedron. This behavior of the four a's is imitated by the b's, c's, d's and e's, so that altogether we have all the even permutations of the five letters: the rotation group of the icosahedron (or of the dodecahedron) is isomorphic to the alternating group A_5. The 60 rotations may be counted as follows: 4 pentagonal rotations about each of 6 axes, 2 trigonal rotations about each of 10 axes, 1 half-turn about each of 15 axes, and the identity [Coxeter **1**, p. 50].

We shall find that the above list exhausts the finite groups of rotations. As a first step in this direction, we observe that all the axes of rotation must pass through a fixed point. In fact, we can just as easily prove a stronger result:

15.41 *Every finite group of isometries leaves at least one point invariant.*

Proof. A finite group of isometries transforms any given point into a finite set of points, and transforms the whole set of points into itself. This, like any finite (or bounded) set of points, determines *a unique smallest sphere that contains all the points on its surface or inside*: unique because, if there were two equal smallest spheres, the points would belong to their common part, which is a "lens"; and the sphere that has the rim of the lens for a great circle is smaller than either of the two equal spheres, contradicting our supposition that these spheres are as small as possible. (The shaded area in Figure 15.4*d* is a section of the lens.) The group transforms this unique sphere into itself. Its surface contains some of the points, and therefore all of them. Its center is the desired invariant point.

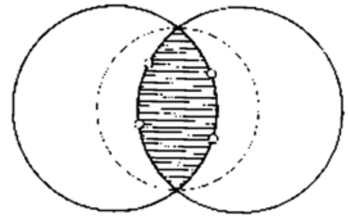


Figure 15.4d

It follows that any finite group of rotations may be regarded as operating on the surface of a sphere. In such a group G, each rotation, other than the identity, leaves just two points invariant, namely the *poles* where the axis of rotation intersects the sphere. A pole P is said to be p-gonal ($p \geqslant 2$) if it belongs to a rotation of period p. The p rotations about P, through various multiples of the angle $2\pi/p$, are those rotations of G which leave P invariant. Any other rotation of G transforms P into an "equivalent" pole, which is likewise p-gonal. Thus all the poles fall into sets of equivalent poles. All the poles in a set have the same period p, but two poles of the same period do not necessarily belong to the same set; they belong to the same set only if one is transformed into the other by a rotation that belongs to G.

Any set of equivalent p-gonal poles consists of exactly n/p poles, where n is the order of G. To prove this, take a point Q on the sphere, arbitrarily near to a pole P belonging to the set. The p rotations about P transform Q into a small p-gon round P. The other rotations of G transform this p-gon into congruent p-gons round all the other poles in the set. But the n rotations of G transform Q into just n points (including Q itself). Since these n points are distributed into p-gons round the poles, the number of poles in the set must be n/p.

The $n - 1$ rotations of G, other than the identity, consist of $p - 1$ for each p-gonal axis, that is, $\frac{1}{2}(p - 1)$ for each p-gonal pole, or

$$\tfrac{1}{2}(p - 1)n/p$$

for each set of n/p equivalent poles. Hence

$$n - 1 = \tfrac{1}{2}n \sum (p - 1)/p,$$

where the summation is over the sets of poles. This equation may be expressed as

$$2 - \frac{2}{n} = \sum \left(1 - \frac{1}{p}\right).$$

If $n = 1$, so that G consists of the identity alone, there are no poles, and the sum on the right has no term. In all other cases $n \geqslant 2$, and therefore

$$1 \leqslant 2 - \frac{2}{n} < 2.$$

It follows that the number of sets of poles can only be 2 or 3; for, the single term $1 - 1/p$ would be less than 1, and the sum of 4 or more terms would be

$$\geqslant 4(1 - \tfrac{1}{2}) = 2.$$

If there are 2 sets of poles, we have

$$2 - \frac{2}{n} = 1 - \frac{1}{p_1} + 1 - \frac{1}{p_2},$$

that is,

$$\frac{n}{p_1} + \frac{n}{p_2} = 2.$$

But two positive integers can have the sum 2 only if each equals 1; thus

$$p_1 = p_2 = n,$$

each of the 2 sets of poles consists of one n-gonal pole, and we have the cyclic group C_n with a pole at each end of its single axis.

Finally, in the case of 3 sets of poles we have

$$2 - \frac{2}{n} = 1 - \frac{1}{p_1} + 1 - \frac{1}{p_2} + 1 - \frac{1}{p_3},$$

whence

15.42
$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1 + \frac{2}{n}.$$

Since this is greater than $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$, the three periods p_i cannot all be 3 or more. Hence at least one of them is 2, say $p_3 = 2$, and we have

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2} + \frac{2}{n},$$

whence
$$(p_1 - 2)(p_2 - 2) = 4(1 - p_1p_2/n) < 4$$

(cf. 10.33), so that the only possibilities (with $p_1 \leqslant p_2$ for convenience) are:

$$p_1 = 2,\ p_2 = p,\ n = 2p; \qquad p_1 = 3,\ p_2 = 3,\ n = 12;$$
$$p_1 = 3,\ p_2 = 4,\ n = 24; \qquad p_1 = 3,\ p_2 = 5,\ n = 60.$$

We recognize these as the dihedral, tetrahedral, octahedral and icosahedral groups.

This completes our proof [Klein **3,** p. 129] that

15.43 *The only finite groups of rotations in three dimensions are the cyclic groups C_p ($p = 1, 2, \ldots$), the dihedral groups D_p ($p = 2, 3, \ldots$), the tetrahedral group A_4, the octahedral group S_4, and the icosahedral group A_5.*

(To avoid repetition, we have excluded D_1 which, when considered as a group of rotations, is not only abstractly but geometrically identical with C_2.)

Any solid having one of these groups for its complete symmetry group

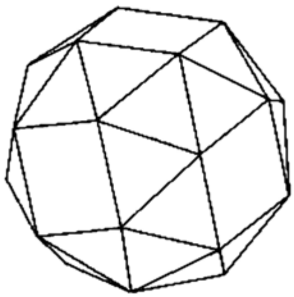


Figure 15.4e

(such as the Archimedean *snub cube** shown in Figure 15.4*e*, whose group is S_4) can occur in two *enantiomorphous* varieties, *dextro* and *laevo* (i.e., right- and left-handed): mirror images that cannot be superposed by a continuous motion.

EXERCISES

1. Interpret the following permutations as rotations of the octahedron (Figure 15.4*b*):

$$(a\,b\,c\,d),\quad (a\,b\,c),\quad (a\,b),\quad (a\,b)(c\,d).$$

Count the rotations of each type, and check with the known order of S_4.

2. Using the symbol (p_1, p_2, p_3) for the group having three sets of poles of periods p_1, p_2, p_3, consider the possibility of stretching the notation so as to allow $(1, p, p) \cong C_p$ as well as

$$(2, 2, p) \cong D_p, \qquad (2, 3, 3) \cong A_4,$$
$$(2, 3, 4) \cong S_4, \qquad (2, 3, 5) \cong A_5.$$

15.5 FINITE GROUPS OF ISOMETRIES

Having enumerated the finite groups of rotations, we can easily solve the wider problem of enumerating the finite groups of isometries (cf. § 2.7). Since every such group leaves one point invariant, we are concerned only with isometries having fixed points. Such an isometry is a rotation or a rotatory inversion according as it is direct or opposite (7.15, 7.41).

If a finite group of isometries consists entirely of rotations, it is one of the groups G considered in § 15.4. If not, it contains such a group G as a subgroup of index 2, that is, it is a group of order $2n$ consisting of n rotations $\mathrm{S}_1, \mathrm{S}_2, \ldots, \mathrm{S}_n$ and an equal number of rotatory inversions T_1,

* The vertices of the snub cube constitute a distribution of 24 points on a sphere for which the smallest distance between any 2 is as great as possible. This was conjectured by K. Schütte and B. L. van der Waerden (*Mathematische Annalen,* **123** (1951), pp. 108, 123) and was proved by R. M. Robinson (*ibid.,* **144** (1961), pp. 17–48). The analogous distribution of 6 or 12 points is achieved by the vertices of an octahedron or an icosahedron, respectively. For 8 points the figure is not, as we might at first expect, a cube, but a square antiprism [Fejes Tóth **1**, pp. 162–164].

$T_2, \ldots, T_n$. For, if the group consists of n rotations S_i and (say) m rotatory inversions T_i, we can multiply by T_1 so as to express the same $n + m$ isometries as S_iT_1 and T_iT_1. The n isometries S_iT_1, being rotatory inversions, are the same as T_i (suitably rearranged if necessary), and the m isometries T_iT_1, being rotations, are the same as S_i. Therefore $m = n$.

If the central inversion I belongs to the group, the n rotatory inversions are simply

$$S_iI = IS_i \quad (i = 1, 2, \ldots, n),$$

and the group is the direct product $G \times \{I\}$, where G is the subgroup consisting of the S's and $\{I\}$ denotes the group of order 2 generated by I. (As an abstract group, $\{I\}$ is, of course, the same as C_2 or D_1.)

If I does not belong, the $2n$ transformations S_i and T_iI form a group of rotations of order $2n$ which has the same multiplication table as the given group consisting of S_i and T_i. For, if $S_iT_j = T_k$,

$$S_iT_jI = T_kI,$$

and if $T_iT_j = S_k$,

$$T_iIT_jI = T_iI^2T_j = T_iT_j = S_k.$$

In other words, a group of n rotations and n rotatory inversions, not including I, is isomorphic to a rotation group G' of order $2n$ which has a subgroup G of order n. To complete our enumeration, we merely have to seek such pairs of related rotation groups. Each pair yields a "mixed" group, say $G'G$, consisting of all the rotations in the smaller group G, along with the remaining rotations in G' each multiplied by the central inversion I. Looking back at § 15.4, we see that the possible pairs are

$$C_{2n}C_n, \quad D_nC_n, \quad D_nD_{\frac{1}{2}n}\,(n \text{ even}), \quad S_4A_4.$$

Thus we can complete Table III on p. 413.

EXERCISES

1. Determine the symmetry groups of the following figures: (a) an orthoscheme $O_0O_1O_2O_3$ (Figure 10.4c) with $O_0O_1 = O_2O_3$; (b) an n-gonal antiprism (n even or odd).

2. Designate in the $G'G$ notation the direct product of the group of order 3 generated by a rotation about a vertical axis and the group of order 2 generated by the reflection in a horizontal plane.

15.6 GEOMETRICAL CRYSTALLOGRAPHY

> *The sense in which a snail's shell winds is an inheritable character founded in its genetic constitution, as is . . . the winding of the intestinal duct in the species Homo sapiens. . . . Also the deeper chemical constitution of our human body shows that we have a screw, a screw that is turning the same way in every one of us.* . . . A horrid manifestation of this genotypical asymmetry is a metabolic disease called phenylketonuria, leading to insanity, that man contracts when a small quantity of laevo-phenylalanine is added to his food, while the dextro- form has no such disastrous effects.*
>
> H. Weyl [**1**, p. 30]

The discussion of symmetry groups has been phrased in such a way as to be valid not only in Euclidean space but in absolute space. However, it seems appropriate to mention the application of these ideas to the practical science of crystallography. Accordingly, in this digression the geometry is strictly Euclidean.

Crystallographers are interested in those finite groups of isometries which arise as subgroups (and factor groups) of symmetry groups of three-dimensional lattices. By § 4.5, these are the special cases in which the only rotations that occur have periods 2, 3, 4 or 6. This crystallographic restriction reduces the rotation groups to

$$C_1,\ C_2,\ C_3,\ C_4,\ C_6,\ D_2,\ D_3,\ D_4,\ D_6,\ A_4,\ S_4,$$

the direct products to these eleven each multipled by $\{\mathrm{I}\}$, and the mixed groups to

$$C_2C_1,\ C_4C_2,\ C_6C_3,\ D_2C_2,\ D_3C_3,\ D_4C_4,\ D_6C_6,\ D_4D_2,\ D_6D_3,\ S_4A_4.$$

(Of course, $C_1 \times \{\mathrm{I}\}$ is just $\{\mathrm{I}\}$ itself.)

These 32 groups are called the *crystallographic point groups* or "*crystal classes.*" Every crystal has one of them for its symmetry group, and every group except C_6C_3 occurs in at least one known mineral. In the more familiar notation of Schoenflies [see, e.g., Burckhardt **1**, p. 71], the groups are respectively

$$\begin{gathered}C_1,\ C_2,\ C_3,\ C_4,\ C_6,\ D_2,\ D_3,\ D_4,\ D_6,\ T,\ O,\\ C_i,\ C_{2h},\ C_{3i},\ C_{4h},\ C_{6h},\ D_{2h},\ D_{3d},\ D_{4h},\ D_{6h},\ T_h,\ O_h,\\ C_s,\ S_4,\ C_{3h},\ C_{2v},\ C_{3v},\ C_{4v},\ C_{6v},\ D_{2d},\ D_{3h},\ T_d.\end{gathered}$$

To avoid possible confusion, observe that our C_4C_2 and S_4 ("S" for "symmetric") are Schoenflies's S_4 and O (for "octahedral"). The 32 groups are customarily divided into seven *crystal systems*, as follows:

Triclinic:	C_1,	$\{\mathrm{I}\}$.	
Monoclinic:	C_2, $C_2 \times \{\mathrm{I}\}$, C_2C_1.		
Orthorhombic:			D_2, $D_2 \times \{\mathrm{I}\}$, D_2C_2.

Rhombohedral:	C_3,	$C_3 \times \{I\}$,		D_3,	$D_3 \times \{I\}$,	D_3C_3.	
Tetragonal:	C_4,	$C_4 \times \{I\}$,	C_4C_2,	D_4,	$D_4 \times \{I\}$,	D_4C_4,	D_4D_2.
Hexagonal:	C_6,	$C_6 \times \{I\}$,	C_6C_3,	D_6,	$D_6 \times \{I\}$,	D_6C_6,	D_6D_3.
Cubic:	A_4,	$A_4 \times \{I\}$,		S_4,	$S_4 \times \{I\}$,	S_4A_4.	

Table I (on p. 413) is a complete list of the 17 discrete groups of isometries in two dimensions involving two independent translations. The analogous groups in three dimensions are the discrete groups of isometries involving three independent translations. The enumeration of these *space groups* is the central problem of mathematical crystallography. The complete list contains $65 + 165 = 230$ groups.

The first 65 are composed entirely of *direct* isometries. Although these were enumerated as long ago as 1869 by C. Jordan [see Hilton **1**, p. 258], they are usually attributed to L. Sohncke who, in 1879, pointed out their application to crystallography. The most obvious group consists of translations alone. The remaining 64 of the 65 contain also rotations and screw displacements; 22 of them occur in 11 enantiomorphous pairs which are mirror images of each other (one containing right-handed screw displacements and the other the reflected left-handed screw displacements). This explains the phenomenon of optical activity [Sayers and Eustace **1**, pp. 238–241, 248–252]. From the standpoint of pure geometry or pure group theory, it would be more natural to ignore this distinction of sense, thus reducing the number 65 to 54, and the total of 230 to 219 [Burckhardt **1**, p. 161].

The remaining 165 groups contain not only direct but also *opposite* isometries: reflections, rotatory reflections (or rotatory inversions), and glide reflections. Their enumeration, by Fedorov in Russia (1890), Schoenflies in Germany (1891), and Barlow in England (1894), provides one of the most striking instances of independent discovery in different places using different methods. Fedorov, who obtained the 230 as $73 + 54 + 103$ instead of $65 + 165$, was probably unaware of the preliminary work of Jordan and Sohncke. It is quite certain that Schoenflies knew nothing of Fedorov, and that Barlow's work was independent of both.

EXERCISE

Determine the symmetry groups of the following figures: (a) a rectangular parallelepiped (e.g., a brick), (b) a rhombohedron; (c) a regular dodecahedron with an inscribed cube (whose 8 vertices occur among the 20 vertices of the dodecahedron).

15.7 THE POLYHEDRAL KALEIDOSCOPE

In combining three reflections . . . the effect is highly pleasing.

Sir David Brewster (1781–1848)

[Brewster **1**, p. 93]

Table III (on p. 413) is a complete list of the finite groups of isometries. In the preceding section, we selected from this list those groups which satisfy

the crystallographic restriction. Another significant way to make a selection (partly overlapping with the previous way) is to pick out those groups which are generated by reflections, namely,

$$D_nC_n\ (n \geqslant 1), \quad D_{2n}D_n\ (n \text{ odd}), \quad D_n \times \{\mathrm{I}\}\ (n \text{ even}),$$
$$S_4A_4, \qquad S_4 \times \{\mathrm{I}\}, \qquad A_5 \times \{\mathrm{I}\}.$$

(We have now returned to absolute geometry!)

D_1C_1 (Schoenflies's C_s, which we previously denoted by C_2C_1) is the group of order 2 generated by a single reflection. D_2C_2 or D_2D_1 (Schoenflies's C_{2v}) is the group of order 4 generated by two orthogonal reflections. The remaining groups D_nC_n are the symmetry groups of the n-gonal pyramids. In other words, these are the groups D_n of § 2.7 in a different notation. (We now reserve the symbol D_n for the dihedral group of *rotations*, which is, of course, isomorphic to D_nC_n. Weyl [**1**, p. 80] makes the distinction by calling the rotation group D'_n and the mixed group D'_nC_n.)

$D_2 \times \{\mathrm{I}\}$ is a group of order 8 (abstractly $C_2 \times C_2 \times C_2$) generated by three orthogonal reflections. The remaining groups $D_{2n}D_n$ (n odd) and $D_n \times \{\mathrm{I}\}$ (n even) are the symmetry groups of the n-gonal prisms, or of their reciprocals, the dipyramids.

S_4A_4, the symmetry group of the regular tetrahedron, is derived from the rotation group A_4 by adjoining reflections, such as the reflection in the plane $ABA'B'$ (Figure 10.5a) which joins the edge AB to the midpoint of the opposite edge CD. (The product of this reflection and the central inversion is the half-turn about the join of the midpoints of the two opposite edges CD', $C'D$ of the cube. This half-turn, which interchanges the two reciprocal tetrahedra $ABCD$, $A'B'C'D'$, is one of the twelve rotations in S_4 that do not belong to the subgroup A_4; thus it illustrates our special meaning for the "mixed" symbol S_4A_4.) Since the remaining Platonic solids are centrally symmetrical, their symmetry groups are simply $S_4 \times \{\mathrm{I}\}$ and $A_5 \times \{\mathrm{I}\}$.

For a practical demonstration in Euclidean space, take the two hinged mirrors of § 2.7, inclined at $180°/n$, which demonstrate the group D_nC_n. Standing them upright on a separate horizontal mirror, we obtain the symmetry group of the n-gonal prism, i.e., the direct product of D_nC_n and the group of order 2 generated by the horizontal reflection. To demonstrate the three remaining groups, remove the third mirror, and let the first two stand vertically on the table at an angle of 60°, as in the demonstration of D_3C_3. Now hold the third mirror obliquely, with its horizontal edge l on the table top at right angles to one of the vertical mirrors and touching the front lower corner of the other. Gradually rotating this third mirror about its edge l from an almost horizontal position (by raising its nearer edge, opposite to l), we observe at a certain stage two faces of a regular tetrahedron $\{3, 3\}$. Each face is subdivided into six right-angled triangles, one of which is actually the exposed portion of the table top. At a later stage we see three faces of an octahedron $\{3, 4\}$; still later, four faces of an icosahedron $\{3, 5\}$. Finally, when the adjustable mirror is vertical like the others, we see a theoretically infinite number of faces of the regular tessellation $\{3, 6\}$, subdivided in the manner of Figure 4.6d. This device, employing ordinary rectangular mirrors, is a simplified version of Möbius's trihedral kaleidoscope in which the three mirrors are cut in the shape of suitable sectors of a circle [Coxeter **1**, p. 83].

When the E edges of the general Platonic solid $\{p, q\}$ are projected from

its center onto a concentric sphere, they become E arcs of great circles, decomposing the surface into F regions which are "spherical p-gons." In this manner the polyhedron yields a "spherical tessellation" which closely resembles the plane tessellation of § 4.6. The symmetry group of $\{p, q\}$ is derived from the symmetry group of one face by adding the reflection in a side of that face. Thus it is generated by reflections in the sides of a spherical triangle whose angles are π/p (at the center of a face), $\pi/2$ (at the midpoint of an edge), and π/q (at a vertex). This spherical triangle is a fundamental region for the group, since it is transformed into neighboring regions by the three generating reflections.

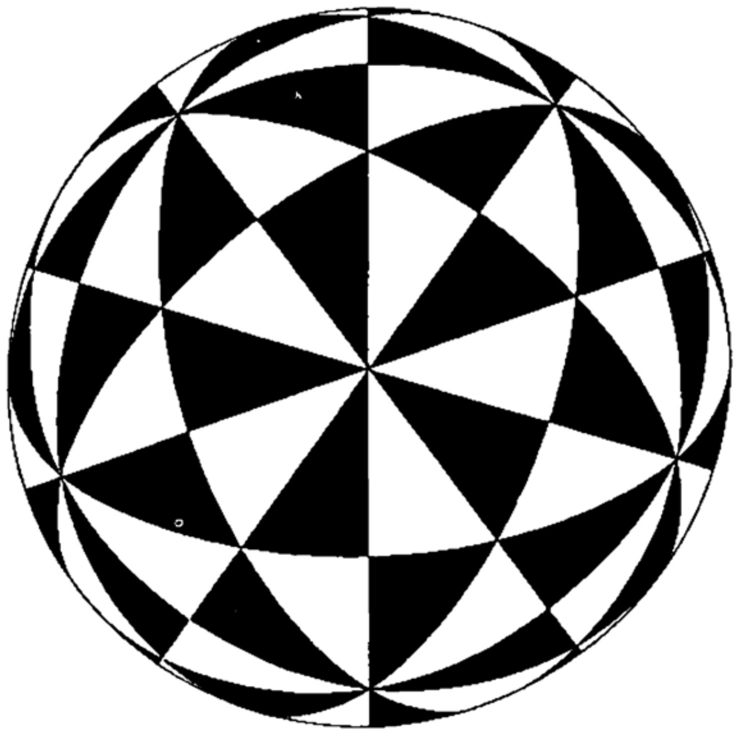


Figure 15.7a

The network of such triangles, filling the surface of the sphere, is cut out by all the planes of symmetry of the polyhedron, namely the planes joining the center to the edges of both $\{p, q\}$ and its reciprocal $\{q, p\}$. In Figure 15.7*a* (where p and q are 3 and 5), alternate regions have been blackened so as to exhibit both the complete symmetry group $A_5 \times \{I\}$ and the rotational subgroup A_5, which preserves the coloring.

Instead of deriving the network of spherical triangles from the regular polyhedron, we may conversely derive the polyhedron from the network. The ten triangles in the middle of Figure 15.7*a* evidently combine to form a face of the blown-up dodecahedron, and the six triangles surrounding a

point where the angles are 60° combine to form a face of the blown-up icosahedron.

EXERCISES

1. Interpret the symbol $\{p, 2\}$ as a spherical tessellation ("dihedron") whose faces consist of two hemispheres, and $\{2, p\}$ as another whose faces consist of p lunes.

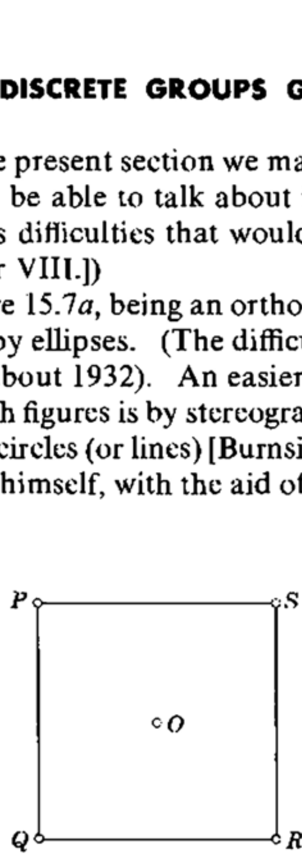
2. How many planes of symmetry does each Platonic solid have? Provided p and q are greater than 2, this number is always a multiple of 3, namely $3c$ in the notation of Ex. 1 at the end of § 10.4.

3. Dividing 4π by the area of the fundamental region, obtain a formula for the order of the symmetry group of $\{p, q\}$. Reconcile this with the formula for E (the number of edges) in 10.32.

15.8 DISCRETE GROUPS GENERATED BY INVERSIONS

In the present section we make one more digression into Euclidean space, so as to be able to talk about inversion. (The absolute theory of inversion presents difficulties that would take us too far afield. [See Sommerville **1**, Chapter VIII.])

Figure 15.7*a*, being an orthogonal projection, represents 10 of the 15 great circles by ellipses. (The difficult task of drawing it was undertaken by J. F. Petrie about 1932). An easier, and perhaps more significant, way to represent such figures is by stereographic projection (§ 6.9), so that the great circles remain circles (or lines) [Burnside **1**, pp. 406–407]. The reader can readily do this for himself, with the aid of the following simple instructions.



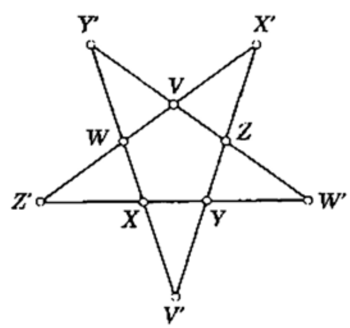


Figure 15.8a

Figure 15.8*a* shows a square $PQRS$ with center O, and a regular pentagon $VWXYZ$ with its sides extended to form a pentagram $V'X'Z'W'Y'$. With radius PQ and centers P, Q, R, S, draw four circles. These, along with two lines through O parallel to the sides of the square, represent 6 great circles, one in each of the 6 planes of symmetry of the tetrahedron $\{3, 3\}$, which are the planes joining pairs of opposite edges of a cube. Adding the cir-

cumcircle and diagonals of the square, we have altogether 9 great circles, one in each of the 9 planes of symmetry of the cube $\{4, 3\}$, which include 3 planes parallel to its faces.

With radius VX' $(=VX)$ and centers V, W, X, Y, Z, draw five circles. With radius VW' $(=V'W')$ and centers V', W', X', Y', Z', draw five more circles. These ten circles, along with the five lines VV', WW', XX', YY', ZZ', represent 15 great circles (Figure 15.7*a*), one in each of the 15 planes of symmetry of the icosahedron $\{3, 5\}$ or of the dodecahedron $\{5, 3\}$. (These planes join pairs of opposite edges of either solid.)

To justify these statements we merely have to examine the curvilinear triangles* and observe that each has angles π/p, π/q, $\pi/2$.

Since stereographic projection is an inversion (Figure 6.9*a*), and since an inversion transforms a reflection into an inversion, the figures so constructed are, in fact, representations of the abstract groups S_4, $S_4 \times C_2$, and $A_5 \times C_2$ as groups generated by inversions. In other words, they are configurations of circles so arranged that the whole figure is symmetrical by inversion in each circle. (Of course, any straight lines that occur are to be regarded as circles of infinite radius. As we saw in § 6.4, inversion in such a "circle" is simply reflection in the line.) Any one of the regions into which the plane is decomposed will serve as a fundamental region, and the generators of the group may be taken to be the inversions in its sides.

For a group generated by just one inversion, we may invert the circle into a straight line so as to obtain the group D_1 of order 2, generated by a single reflection (§ 2.5). The groups generated by inversions in two intersecting circles are essentially the same as the groups D_n of order $2n$, generated by reflections in two intersecting lines (§ 2.7). If the circles of two generating inversions are in contact, they can be inverted into parallel lines, and we have the limiting case D_∞ (Figure 3.7*b*). Two nonintersecting circles can be inverted into concentric circles. Inversions in them generate an infinite sequence of concentric circles whose radii are in geometric progression. Abstractly, the group is again D_∞, but the center is a "point of accumulation" (§ 7.6). So is the point of contact in the case of the group generated by inversions in two touching circles. A group is said to be *discrete* if it has no points of accumulation. Thus, in describing discrete groups generated by inversions, we may insist that every two of the generating circles intersect properly, and do not touch.

For a discrete group generated by three inversions, the fundamental region is a curvilinear triangle whose angles are submultiples of π: say π/p_1, π/p_2, π/p_3. For instance, two radii of a circle, forming an angle π/p, cut out a sector which may be regarded as a "triangle" with angles π/p, $\pi/2$, $\pi/2$; this is a fundamental region for the group $D_p \times D_1$ of order $4p$, generated by reflections in the radii and inversion in the circle. In this case

* For the effect of projecting in a different direction, see Coxeter, *American Mathematical Monthly*, **45** (1938), pp. 523–525, Figs. 4 and 5.

15.81
$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} > 1,$$

so that the angle sum of the triangle is greater than π: an obvious consequence of the fact that the sector is derived from a spherical triangle (see § 6.9) by stereographic projection, which preserves angles. Every solution of the inequality 15.81 (cf. 15.42) is a triangle that can be drawn with great circles on a sphere. We thus obtain again the symmetry groups

$$S_4, \quad S_4 \times C_2, \qquad A_5 \times C_2$$

of the Platonic solids.

When $1/p_1 + 1/p_2 + 1/p_3 = 1$, so that the angle sum is exactly π, we have the infinite "Euclidean" groups **p6m**, **p4m**, **p31m** (see Table I and Figure 4.6*d*). We could transform all the straight lines into circles by means of an arbitrary inversion; but then, since the pattern is infinitely extended, the center of inversion would be a point of accumulation.

When $1/p_1 + 1/p_2 + 1/p_3 < 1$, so that the angle sum of the fundamental region is less than π, we may still take two of the three sides to be straight, but now their point of intersection A is outside the circle q to which the third side belongs, with the result that there is a circle Ω orthogonal to all three (Figure 15.8*b*); the tangents from A to q are radii of Ω.

Since Ω is invariant for each of the generating inversions, it is invariant for the whole group. The circle q decomposes the interior of Ω into two

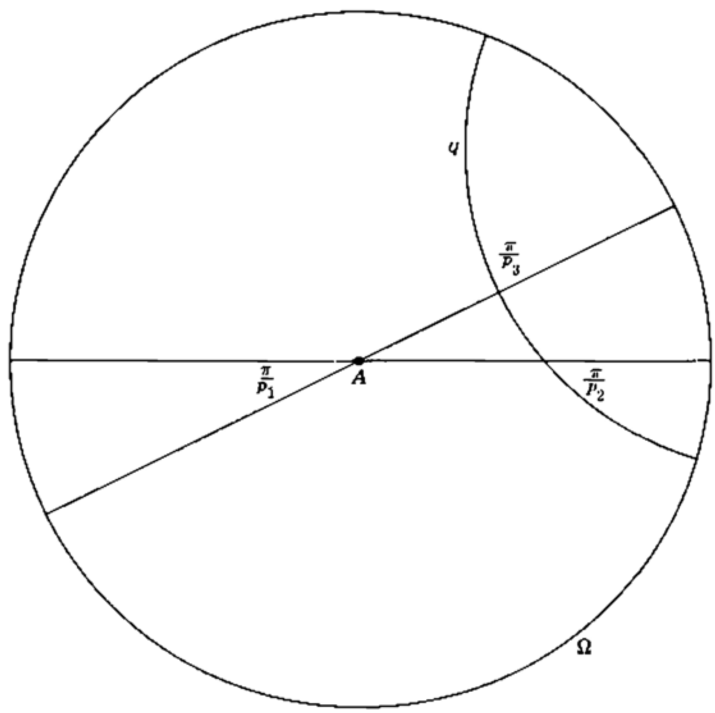


Figure 15.8b

unequal regions and inverts each of these regions into the other. Therefore the number of triangles is the same in both regions. But the larger region includes a replica of the smaller. Hence, by Bolzano's definition of an infinite set (namely, a set that has the same power as a proper subset), the number of triangles is infinite; that is, *the group is infinite.*

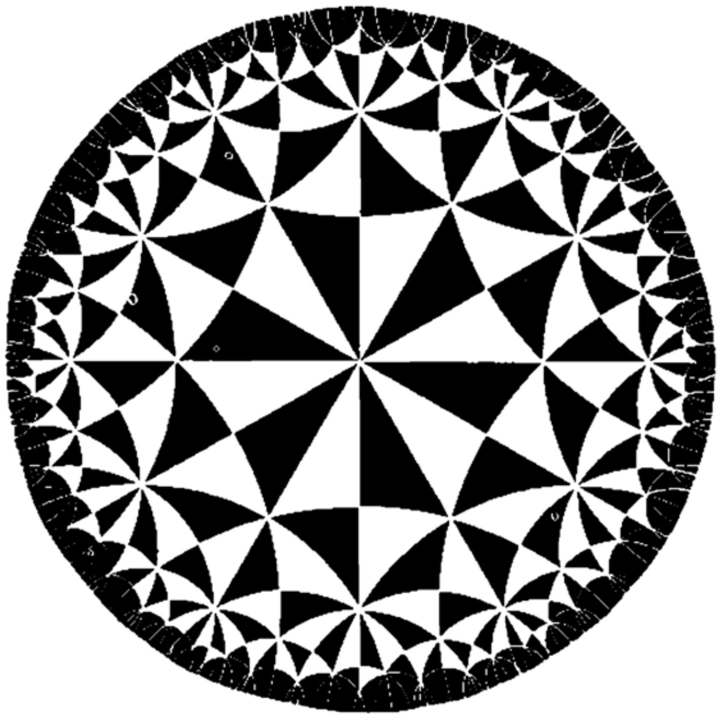


Figure 15.8c

The case when p_1, p_2, p_3 are 6, 4, 2 is shown in Figure 15.8*c*. Unlike Figure 15.7*a*, this is not a picture of a solid object. Our familiarity with three-dimensional space enables us to accept the idea that the triangles in Figure 15.7*a* are all the same size, even though the peripheral ones are made to look smaller by perspective foreshortening. In the case of Figure 15.8*c*, the smaller peripheral triangles are essentially the same shape as those in the middle (since they have the same angles), but we no longer find it easy to imagine that they are, in some sense, the same *size*. In trying to stretch our imagination to this extent, we are taking a first step towards appreciating hyperbolic geometry, which is the subject of our next chapter.

The reader may wonder why we admit such groups as being worthy of consideration, seeing that the circle Ω contains infinitely many points of accumulation. However, when we accept the non-Euclidean standpoint, so that the circles and inversions are regarded as lines and reflections, the consequent distortion of distance makes Ω infinitely far away, so that the points of accumulation disappear.

EXERCISES

1. If a system of concentric circles is transformed into itself by inversion in each circle, the radii are in geometric progression.

2. If three circles form a "triangle" with angles π/p_1, π/p_2, π/p_3, the inversions R_1, R_2, R_3 in its sides satisfy the relations

$$R_1^2 = R_2^2 = R_3^2 = (R_2R_3)^{p_1} = (R_3R_1)^{p_2} = (R_1R_2)^{p_3} = 1.$$

These relations suffice to define the abstract group generated by R_1, R_2, R_3 [Coxeter and Moser **1**, pp. 37, 55].

3. Given an angle π/p_1 at the center A of a circle Ω of unit radius, as in Figure 15.8*b*, find expressions (in terms of p_1 and p_2) for the radius of the circle q and for the distance from A to its center, in the case when $p_3 = 2$.

4. Invert Figure 15.8*c* in a circle whose center lies on Ω; that is, replace the circle Ω by a straight line, so that all the inverting circles have their centers on this line. (Such an arrangement provides an alternative proof that the group is infinite. For if its order is g, the infinite half plane is filled with g curvilinear triangles, each having a finite area!)

5. In Figure 15.8*c*, two of the small triangles (one white and one black) with a common hypotenuse form together a "curvilinear kite" having three right angles and one angle of 60°. Trace part of the figure so as to exhibit a network of such kites, alternately white and black. We now have an instance of a group generated by four inversions. Can it happen that more than four inversions are needed to generate a discrete group?

16
Hyperbolic geometry

Absolute geometry is not *categorical:* it is two geometries in one. To be precise, it leaves open the question of the existence of ultraparallel lines (see the end of § 15.2). In § 16.1 we shall compare the two possible answers, giving the unfamiliar the same status as the familiar. In § 16.2 we shall justify this action by means of a proof of *relative consistency*. Thereafter, casting aside all scruples, we shall plunge wholeheartedly into the "new universe" which Bolyai "created from nothing."

16.1 THE EUCLIDEAN AND HYPERBOLIC AXIOMS OF PARALLELISM

In the author there lives the perfectly purified conviction (such as he expects too from every thoughtful reader) that by the elucidation of this subject one of the most important and brilliant contributions has been made to the real victory of knowledge, to the education of the intelligence, and consequently to the uplifting of the fortunes of men.

J. Bolyai (1802-1860)

[Carslaw **1**, p. 31]

In § 12.6, we mentioned the question whether the two rays parallel to a given line r from an outside point A are, or are not, collinear. By applying a suitable isometry, we see that the answer is independent of the position of r.

It is true, though less obvious, that, for a given r, the answer is independent of the position of A. Suppose, if possible, that the rays parallel to r from A are the two halves of a line q while the rays parallel to r from another point A' form an angle, as in Figure 16.1a. By the transitivity of parallelism, these rays from A' are parallel to q and also to the infinite sequence of parallel lines derived from q and r by applying the group D_∞ generated by reflections in q and r (Figure 3.7b). We obtain a manifest absurdity by considering any one of these lines that lies beyond A' (i.e., in such a position that A' lies between that line and r). (Strictly, this argument makes use of the so-called Axiom of Archimedes, 13.31, which is a consequence of 12.51.)

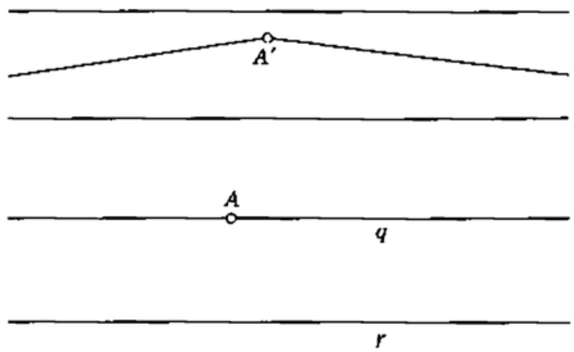


Figure 16.1a

Thus we have a clear-cut distinction between two kinds of geometry, called *Euclidean* and *hyperbolic*, which are derived from absolute geometry by adding just one of the following two alternative axioms:

THE EUCLIDEAN AXIOM. *For some point A and some line r, not through A, there is not more than one line through A, in the plane Ar, not meeting r.*

THE HYPERBOLIC AXIOM. *For some point A and some line r, not through A, there is more than one line through A, in the plane Ar, not meeting r.*

EXERCISE

Each of these axioms implies the stronger statement with "some point A and some line r" replaced by "any point A and any line r." The Euclidean axiom, so amended, is equivalent to the celebrated Postulate V (our 1.25). How does Postulate V break down if we assume the hyperbolic axiom?

16.2 THE QUESTION OF CONSISTENCY

What are we to think of the question: Is Euclidean Geometry true? It has no meaning. We might as well ask . . . if Cartesian coordinates are true and polar coordinates false. One geometry cannot be more true than another; it can only be more convenient.

H. Poincaré (1854-1912)

(*Science and Hypothesis*, New York, 1952)

We observe that the Euclidean and hyperbolic axioms differ by just one word: the vital word "not." It is meaningless to ask which of the two geometries is *true*, and practically impossible to decide which provides a more *convenient* basis for describing astronomical space. From the standpoint of pure mathematics, a more important question is whether either axiom is logically *consistent* with the remaining axioms of absolute geometry. Even this is difficult to answer; for according to the philosopher Gödel, there is no internal proof of consistency for a system that includes infinite sets. We have

to be content with *relative* consistency: if Euclidean geometry is free from contradiction, so is hyperbolic geometry, and vice versa. Relative consistency is established by finding in each geometry a *model* of the other.

One Euclidean model of the hyperbolic plane (due to Poincaré) was mentioned in § 15.8. This uses a circle Ω, as in Figure 16.2*a*. Each pair of inverse points represents a hyperbolic point, and each circle orthogonal to Ω represents a hyperbolic line. The two parallels to r from A are simply the circles through A that touch r at its points of intersection with Ω. (These points are the "ends" of r.) We call this a *conformal* model because angles retain their proper values though distances are inevitably distorted.

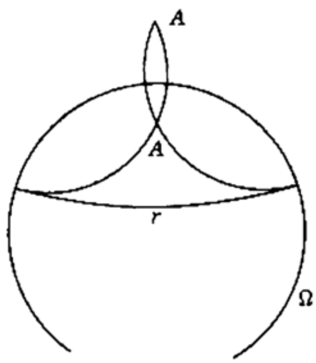


Figure 16.2a

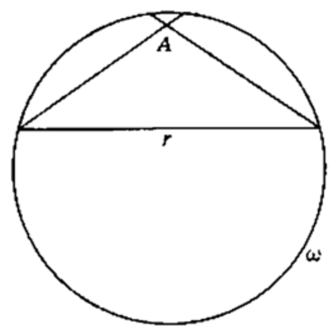


Figure 16.2b

A different Euclidean model, suggested by Beltrami (1835–1900), uses another circle ω, as in Figure 16.2*b*. Each point inside ω represents a hyperbolic point. The two parallels to r from A are the chords joining A to the ends of the chord r. (Chords whose lines intersect outside ω represent ultraparallel lines.) We call this a *projective* model because straight lines remain straight. Nothing is lost if we replace the circle ω in the Euclidean plane by a conic in the projective plane. In fact, much is gained; for it is possible to extend the hyperbolic plane into a projective plane by means of entities defined in the hyperbolic geometry itself [Coxeter **3**, p. 196]. In this way we can prove that hyperbolic geometry is unique or *categorical* [Borsuk and Szmielew **1**, p. 345], unlike absolute geometry, which includes two contrasting possibilities.

When using models, it is desirable to have two rather than one, so as to avoid the temptation to give either of them undue prominence. Our geometric reasoning should all depend on the axioms. The models, having served their purpose of establishing relative consistency [Pedoe **1**, p. 61; Sommerville **1**, pp. 154–159], are no more essential than diagrams.

Klein [**4**, p. 296] exhibited a connection between the conformal and projective models in the manner of Figure 16.2*c*. A sphere, having the same radius as ω, touches the (horizontal) plane at S, the center of both ω and Ω.

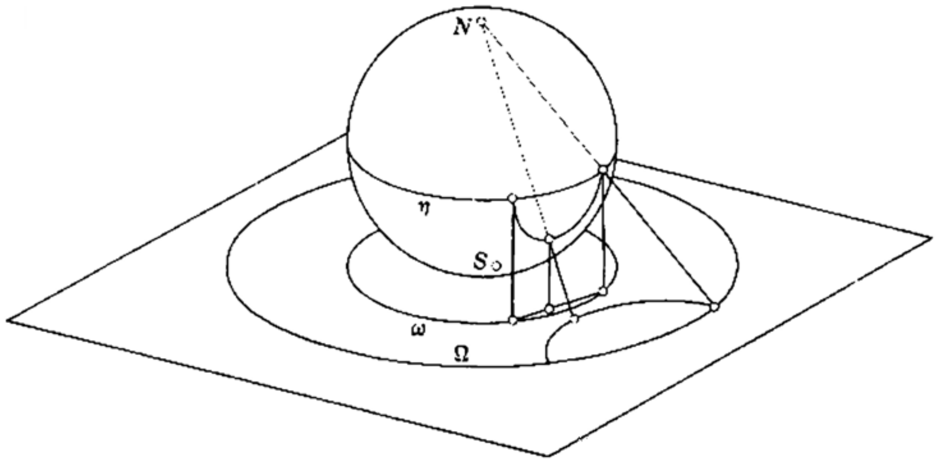


Figure 16.2c

Beginning with the projective model, we use orthogonal (vertical) projection to map ω on the "equator" η of the sphere, and each interior point on two points: one in the southern hemisphere and another (not shown) in the northern hemisphere. Every chord of ω yields a circle in a vertical plane, that is, a circle orthogonal to η. We now map the sphere back into the plane by stereographic projection, so that η projects into the larger circle Ω, concentric with ω. Because of the angle-preserving and circle-preserving nature of stereographic projection, the vertical circles yield horizontal circles orthogonal to Ω, and we have the conformal model.

Instead of stereographic projection onto the tangent plane at the "south pole" S (i.e., inversion with respect to a sphere of radius NS), we could have used stereographic projection (from the same "north pole" N) onto the equatorial plane (i.e., inversion with respect to a sphere through η) so as to make both ω and Ω coincide with η [Coxeter **3**, p. 260]. Klein's procedure is justified by its property of making the two models agree in the immediate vicinity of S. This must have seemed to him more important than making them agree "at infinity."

It must be remembered that both models are in one respect misleading: they give us the impression that the center S should play a special role, whereas, in the abstract hyperbolic plane, all points are alike.

For the sake of completeness, we should mention the problem that the inhabitants of a hyperbolic world would face in trying to visualize the Euclidean plane. One solution [Coxeter **3**, pp. 197–198] is that they could represent the Euclidean points and lines by the lines and planes parallel to a given ray in hyperbolic space!

EXERCISES

1. Reflection in a line of the hyperbolic plane appears, in the conformal model, as inversion with respect to a circle, and in the projective model as a harmonic homology. What is the corresponding transformation in the space of Klein's sphere?

2. Circles appear as circles (not meeting Ω) in the conformal model, and therefore as circles on the sphere (say, in the southern hemisphere) and as ellipses in the projective model.

16.3 THE ANGLE OF PARALLELISM

. . . a sea-change into something rich and strange.

W. Shakespeare (1564-1616)

(*The Tempest*, Act I, Scene 2)

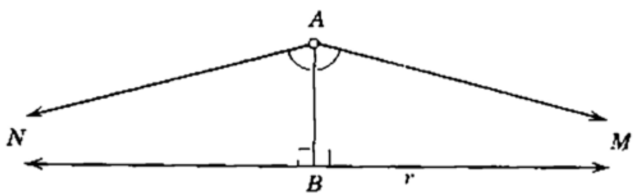


Figure 16.3a

For the rest of this chapter the geometry will be hyperbolic, that is, we shall assume the hyperbolic axiom, which implies that, for any point A and line r, not through A, the two parallels form an angle NAM, as in Figure 16.3a. From A draw AB perpendicular to r. Reflection in AB shows that $\angle BAM$ and $\angle NAB$ are equal acute angles. Following Lobachevsky, we call either of them the *angle of parallelism* corresponding to the distance AB, and write

$$\angle BAM = \Pi(AB).$$

Before we can prove that this function is monotonic, we need a few more properties of asymptotic triangles. While proving 15.26 we discovered that, if a transversal (AD in Figure 15.2f) meets two lines in such a way that the "alternate" angles are equal, then the two lines are ultraparallel. Hence [Carslaw **1**, p. 48]:

16.31 *In an asymptotic triangle EFM, the external angle at E (or F) is greater than the internal angle at F (or E).*

In other words, the sum of the angles of an asymptotic triangle is less than π. This will enable us to prove a kind of converse for Theorem 15.25, to the effect that an asymptotic triangle is determined by its two positive angles:

16.32 *If two asymptotic triangles AEM, $A'E'M'$ have $A = A'$ and $E = E'$, then $AE = A'E'$.*

Proof [Carslaw **1**, p. 50]. If AE and $A'E'$ are not equal, one of them must

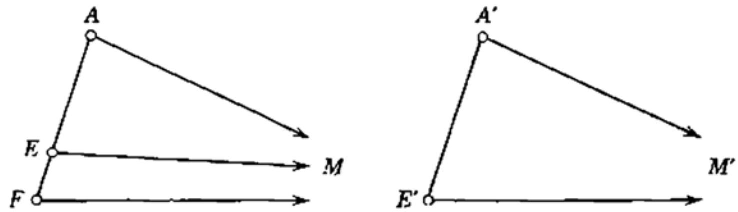


Figure 16.3b

be the greater; let it be $A'E'$, as in Figure 16.3*b*. On E/A, take F so that $AF = A'E'$, and draw FM parallel to AM. By 16.31 and 15.25, we have

$$\angle MEA > \angle MFA = \angle M'E'A' = \angle MEA,$$

which is absurd.

These results will enable us to establish the existence of *a common parallel to two given rays* forming an angle NOM, that is, a line MN which is parallel to OM at one end and to ON at the other. From the given rays OM, ON, cut off any two equal segments OA, OA', as in Figure 16.3*c*. Draw $A'M$ parallel to OM, and AN parallel to ON. Bisect the angles NAM and $NA'M$ by lines a and a'. We shall prove that *these lines are ultraparallel,* and that *the desired common parallel MN is perpendicular to both of them.*

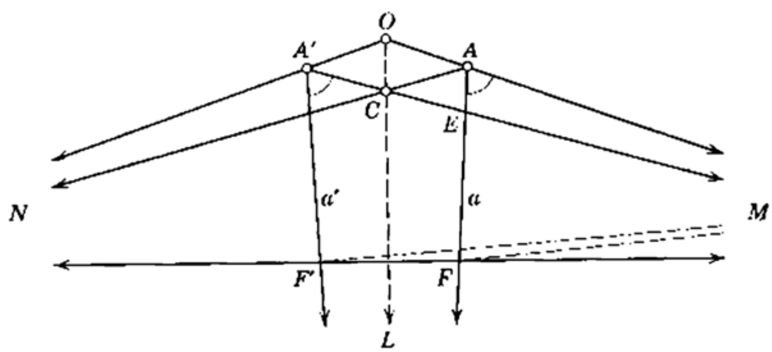


Figure 16.3c

Let $A'M$ meet AN in C, and a in E. Since the whole figure is symmetrical by reflection in OC, the two angles at A and the two angles at A' are all equal.

If possible, let a and a' have a common point L, which is, of course, equidistant from A and A'. Applying 15.25 to the congruent asymptotic triangles ALM and $A'LM$, we deduce that $\angle MLA = \angle MLA'$, which is absurd.

If possible, let a and a' be parallel, with a common end L. Applying 16.32 to the congruent asymptotic triangles AEM and $A'EL$, we deduce that $AE = A'E$, whence E coincides with C, which is absurd.

We conclude that a and a' are ultraparallel. By 15.26, they have a com-

mon perpendicular FF'. Applying 15.25 to the congruent asymptotic triangles AFM and $A'F'M$, we conclude that

$$\angle MFA = \angle MF'A'.$$

If $F'F$ were not parallel to OM, we would have an asymptotic triangle $FF'M$ whose angle sum is π, contradicting 16.31. Hence, in fact, $F'F$ is parallel to OM, and similarly FF' to ON; that is, the line FF' is a common parallel to the two rays as desired.

Moreover, this common parallel is *unique,* since two such would be parallel to each other at both ends, contradicting the "clear-cut distinction" between the Euclidean and hyperbolic properties of parallelism (Figure 16.1a). It follows that

16.33 *Any two ultraparallel lines have a unique common perpendicular.*

For, given a and a', we can reconstruct Figure 16.3c as follows: draw any common perpendicular FF', take O on its perpendicular bisector, and let the two parallels through O to the line FF' meet a in A, a' in A'.

For the sake of brevity, we have been content to assert the *existence* of a line through a given point parallel to a given ray, and of a common perpendicular to two given ultraparallel lines. Actual "ruler and compasses" constructions for these lines have been given by Bolyai and Hilbert, respectively [see Coxeter **3**, pp. 204, 191]. Hilbert apparently failed to notice that his construction for the common parallel to AM and $A'N$ remains valid if these lines meet in a point that is not equidistant from A and A', or even if they do not meet at all. In fact [Carslaw **1**, p. 76],

16.34 *Any two nonparallel rays have a unique common parallel.*

This result justifies our use of *ends* as if they were ordinary points: any two ends, M and N, determine a unique line MN.

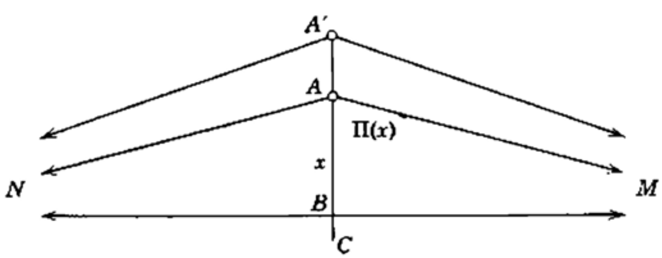


Figure 16.3d

The line through A parallel to BM (Figure 16.3a or d) determines the angle of parallelism $\Pi(AB)$. Conversely, we can now find a distance x whose angle of parallelism $\Pi(x)$ is equal to any given acute angle [Carslaw **1**, p. 77]. In other words, given an acute angle CAM, *we can find a line BM which is both perpendicular to AC and parallel to AM.* We merely have

to reflect AM in AC, obtaining AN, and then draw the common parallel MN, which meets AC in the desired point B. Incidentally, since we can draw through any point a ray parallel to a given ray, it follows that

16.35 *For any two nonperpendicular lines we can find a line which is perpendicular to one and parallel to the other.*

If A' is on the ray A/B, so that $A'B > AB$ (as in Figure 16.3*d*), then

$$\Pi(A'B) < \Pi(AB).$$

(This is simply 16.31, applied to the asymptotic triangle $AA'M$.) It follows that the function $\Pi(x)$ decreases steadily from $\frac{1}{2}\pi$ to 0 when x increases from 0 to ∞.

We naturally call AMN a *doubly asymptotic triangle* [Coxeter **3**, p. 188]. We have seen that such a "triangle" is determined by its one positive angle; in other words,

16.36 *Two doubly asymptotic triangles are congruent if they have equal angles.*

Applying 16.34 to rays belonging to two parallel lines LM, LN, we obtain a third line parallel to both, forming a *trebly asymptotic triangle* LMN. In view of Bolyai's remark 15.24, we may regard such a triangle as a doubly asymptotic triangle whose angle is zero. Accordingly, we shall not be surprised to find that

16.37 *Any two trebly asymptotic triangles are congruent.*

Proof (due to D. W. Crowe). Given any two trebly asymptotic triangles, dissect each into two right-angled doubly asymptotic triangles by drawing an *altitude* (perpendicular to one side and parallel to another, as in 16.35). By 16.36, all the four doubly asymptotic triangles are congruent. Therefore the two trebly asymptotic triangles must be congruent.

EXERCISES

1. Draw figures for Theorems 16.33–16.35 in terms of the conformal and projective models.

2. If a quadrangle $ABED$ has right angles at D and E while $AD = BE$, then the angles at A and B are equal acute angles. (*Hint*: Draw AM and BM parallel to D/E; apply 16.31 to the asymptotic triangle ABM.)

3. The sum of the angles of any triangle is less than two right angles. (*Hint*: For a given triangle ABC, draw AD, BE, CF perpendicular to the line joining the midpoints of BC and CA.)

4. Given an asymptotic triangle ABM with acute angles at both A and B, draw AD perpendicular to BM, and BE perpendicular to AM, meeting in H. Draw HF perpendicular to AB. Then FH is parallel to AM [Bonola **1**, p. 106]. What happens if we deal similarly with rays through A and B which are not parallel but ultraparallel?

5. If two trebly asymptotic triangles have a common side, by what isometry are

they related? (Of course, two trebly asymptotic triangles may have a common side without having a common altitude).

6. The inradius of a trebly asymptotic triangle is the distance whose angle of parallelism is 60°.

7. From any point on a side of a trebly asymptotic triangle, lines drawn perpendicular to the other two sides are themselves perpendicular [Bachmann **1**, p. 222].

16.4 THE FINITENESS OF TRIANGLES

I could be bounded in a nutshell and
count myself a king of infinite space.

W. Shakespeare
(*Hamlet*, Act II, Scene 2)

One of the most elegant passages in the literature on hyperbolic geometry since the time of Lobachevsky is the proof by Liebmann [**1**, p. 43] that the area of a triangle remains finite when all its sides are infinite. C. L. Dodgson (*alias* Lewis Carroll) could not bring himself to accept this theorem; consequently he believed non-Euclidean geometry to be nonsense.

Instead of pursuing a philosophical discussion of the meaning of *area* [Carslaw **1**, pp. 84–90], let us be content to regard it as a numerical function, defined for every simple closed polygon, invariant under isometries, and additive when two polygons are juxtaposed.

Let ABM be any asymptotic triangle. Reflect it in the bisector AF of the angle A to obtain AA_1N, as in Figure 16.4*a*, F being the point where the bisector meets the common parallel MN. Reflect the line BM in the bisector A_1F_1 of $\angle NA_1M$ to obtain A_2N (with A_2 on AM), and then reflect

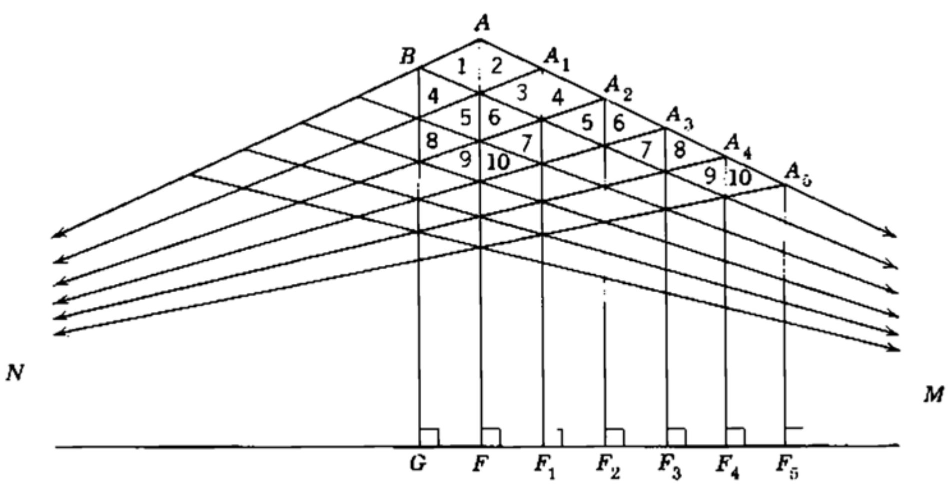


Figure 16.4a

this in AF. Continuing in this manner, we construct a network of triangles whose "vertical" sides bisect the angles at $B, A, A_1, A_2, A_3, \ldots$ and are perpendicular to MN at $G, F, F_1, F_2, F_3, \ldots$. These points are evenly spaced along MN, since they are all derived from F and F_1 by the group D_∞ generated by reflections in AF and A_1F_1; for instance, G is the image of F_1 in the mirror AF. The numbered triangles which fit together to fill the asymptotic triangle ABM are respectively congruent to those which fit together within the finite pentagon $ABGF_1A_1$; in fact, any two triangles that are numbered alike are related by some power of the translation from G to F_1 (or from F to F_2). Hence the area of the asymptotic triangle is less than or equal to the area of the pentagon:

16.41 *Any asymptotic triangle has a finite area.*

Since any doubly asymptotic triangle (Figure 16.3*a*) can be dissected into two asymptotic triangles, it follows that

16.42 *Any doubly asymptotic triangle has a finite area.*

By 16.36, the area of a doubly asymptotic triangle is a function of its angle. Comparing the triangles AMN and $A'MN$ of Figure 16.3*d*, we see that this is a *decreasing* function: the larger triangle has the smaller angle.

Since any trebly asymptotic triangle can be dissected into two doubly asymptotic triangles (as in the proof of 16.37), 16.42 implies

16.43 *Any trebly asymptotic triangle has a finite area.*

By 16.37, this area is a constant, depending only on our chosen unit of measurement.

16.5 AREA AND ANGULAR DEFECT

> *Gauss . . . did not recognize the existence of a logically sound non-Euclidean geometry by intuition or by a flash of genius: . . . on the contrary, he had spent upon this subject many laborious hours before he had overcome the inherited prejudice against it.* [He] *did not let any rumour of his opinions get abroad, being certain that he would be misunderstood. Only to a few trusted friends did he reveal something of his work.*
>
> R. Bonola [**1**, pp. 66-67]

János Bolyai, or Bolyai János (as it is written in Hungarian), announced his discovery of absolute geometry in an appendix to a book by his father, Bolyai Farkas, who was a friend of Gauss. When Gauss saw this book and read the appendix, he wrote a remarkable letter to his old friend, congratulating János and admitting that he himself had thought along the same lines without publishing the results. The original letter (of March 6, 1832) is lost,

but the younger Bolyai's copy of it has been preserved, and it was eventually published in Gauss's collected works [Gauss **1**, vol. 8, pp. 220–225].

This letter contains a wonderful proof that the area of a triangle ABC is proportional to its *angular defect*

$$\pi - A - B - C:$$

the amount by which its angle sum falls short of two right angles. The following paraphrase fills up a few gaps in the argument, while retaining Gauss's systematic division into seven steps, numbered with Roman numerals.

I. *All trebly asymptotic triangles are congruent.* (This is our 16.37.)

II. *The area of a trebly asymptotic triangle has a finite value, say t.* (This is our 16.43.)

III. *The area of a doubly asymptotic triangle AMN is a function of its angle, NAM, say $f(\phi)$, where ϕ is the supplement of this angle.* Given the angle ϕ, we can construct the triangle in a unique fashion (Figure 16.5*a*; cf. 16.3*c*). Gauss used the supplement, rather than the angle NAM itself, to ensure that $f(\phi)$ is an *increasing* function of ϕ. (See the remark after 16.42.)

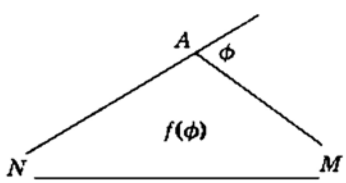


Figure 16.5a

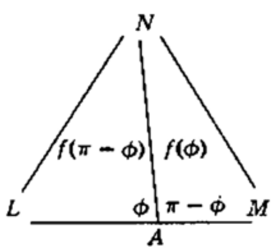


Figure 16.5b

IV. $f(\phi) + f(\pi - \phi) = t$.

This may be seen by fitting together two doubly asymptotic triangles AMN and ANL with supplementary angles, as in Figure 16.5*b*. Here it is understood that $0 < \phi < \pi$. But when ϕ approaches zero, the doubly asymptotic triangle collapses, and when ϕ approaches π it tends to become trebly asymptotic. Hence

16.51 $$f(0) = 0, \qquad f(\pi) = t,$$

and IV is valid for $0 \leqslant \phi \leqslant \pi$.

V. $f(\phi) + f(\psi) + f(\pi - \phi - \psi) = t$.

This, with $\phi > 0, \psi > 0, \phi + \psi < \pi$, may be seen by fitting together three doubly asymptotic triangles whose angles add up to 2π, as in Figure 16.5*c*. It evidently remains valid when ϕ or ψ is zero or $\phi + \psi = \pi$.

VI. $f(\phi) + f(\psi) = f(\phi + \psi)$.

This, with $\phi \geqslant 0, \psi \geqslant 0, \phi + \psi \leqslant \pi$, is obtained algebraically, by writing $\phi + \psi$ instead of ϕ in IV and then using V. It follows that $f(\phi)$ is simply a multiple of ϕ, namely,

16.52 $$f(\phi) = \mu\phi$$

where, by 16.51, $\mu = t/\pi$.

J. H. Lindsay has pointed out that this deduction can be made without assuming the function to be continuous. By VI, with $\phi = \psi$,

$$f(\phi) = \tfrac{1}{2} f(2\phi).$$

Thus 16.52 holds when $\phi = \frac{1}{2}\pi$, again when $\phi = \frac{1}{4}\pi$, and so on; that is, it holds when ϕ is π divided by any power of 2. Appealing again to VI, we deduce that $f(\phi) = \mu\phi$ whenever $\phi = n\pi$, where n is a number which terminates when expressed as a "decimal" in the scale of 2 [cf. Coxeter **3,** p. 102]. For brevity, let us call this a *binary* number.

Suppose, if possible, that, for some particular value of ϕ, $f(\phi) \neq \mu\phi$. Choose a binary number n between the two distinct real numbers ϕ/π and $f(\phi)/\mu\pi$. If $f(\phi) > \mu\phi$, so that

$$\phi < n\pi < \frac{f(\phi)}{\mu},$$

we have, since $f(\phi)$ is an increasing function,

$$f(\phi) < f(n\pi) = \mu n\pi < f(\phi),$$

which is absurd. If, on the other hand, $f(\phi) < \mu\phi$, we can argue the same way with all the inequalities reversed. Hence, in fact, $f(\phi) = \mu\phi$ for all the values of ϕ (from 0 to π).

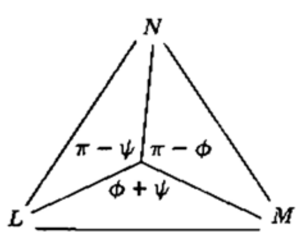


Figure 16.5c

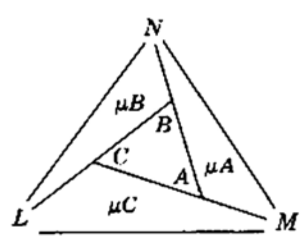


Figure 16.5d

VII. *The area Δ of any triangle ABC (with finite sides) is a constant multiple of its angular defect:*

$$\Delta = \mu(\pi - A - B - C).$$

For this final step, Gauss exhibited ABC as part of a trebly asymptotic triangle by extending its sides in cyclic order, as in Figure 16.5*d*. The re-

maining parts are doubly asymptotic triangles whose areas are μA, μB, μC. Hence

$$\Delta + \mu A + \mu B + \mu C = t = \mu\pi,$$

and the desired formula follows at once.

If we wish, we can follow Lobachevsky in using such a unit of measurement* that the area of a trebly asymptotic triangle is π. Then $\mu = 1$, and the formula is simply

16.53 $$\Delta = \pi - A - B - C.$$

This is strikingly reminiscent of the formula 6.92, which tells us that the area of a spherical triangle drawn on a sphere of radius R is

$$(A + B + C - \pi)R^2.$$

In fact, setting $R^2 = -1$, we find that Gauss's result agrees formally with the area of a triangle drawn on a sphere of radius i. Long before the time of Gauss, it was suggested by J. H. Lambert (1728–1777) that, if a non-Euclidean plane exists, it should resemble a sphere of radius i. This analogy enabled him to derive the formulas of hyperbolic trigonometry (which were later developed rigorously by Lobachevsky) from the classical formulas of spherical trigonometry. Its full significance did not appear till Minkowski (1864–1909) discovered the geometry of space-time, which provided a geometrical basis for Einstein's special theory of relativity. We know now that, in a $(2 + 1)$-dimensional space-time, the hyperbolic plane can be represented without distortion on either sheet of a *sphere of time-like radius*. In the underlying affine space, this kind of sphere is a hyperboloid of two sheets.†

EXERCISES

1. Gauss's formula 16.53 remains valid when the triangle has one or more zero angles.

2. The area of any simple p-gon is equal to its angular defect: the amount by which its angle sum falls short of that of a p-gon in the Euclidean plane. (*Hint:* Dissect the polygon into triangles. Of course, we are now assuming $\mu = 1$.) In Figure 16.4*a*, the area of ABM is equal to that of $ABGF_1A_1$.

3. The product of three translations along the directed sides of a triangle (through the lengths of these sides themselves) is a rotation through the angular defect of the triangle. (These translations are half as long as those in Donkin's theorem, 15.31.) [Lamb **1**, p. 7.]

4. The product of half-turns about the midpoints of the sides of a simple quadrangle (in their natural order) is a rotation through the angular defect of the quadrangle.

5. Any polygon whose angle sum is a submultiple of 2π can be repeated, by half-

* Coxeter, Hyperbolic triangles, *Scripta Mathematica*, **22** (1956), p. 9.

† Coxeter, A geometrical background for de Sitter's world, *American Mathematical Monthly*, **50** (1943), p. 220.

turns about the midpoints of its sides, so as to cover the whole plane without interstices [cf. Somerville **1**, p. 86, Ex. 15]. (*Hint:* See Figures 4.2*b* and *c*.)

16.6 CIRCLES, HOROCYCLES, AND EQUIDISTANT CURVES

A circle is the orthogonal trajectory of a pencil of lines with a real vertex . . . A horocycle is the orthogonal trajectory of a pencil of parallel lines. . . . The orthogonal trajectory of a pencil of lines with an ideal vertex . . . is called an equidistant-curve.

D. M. Y. Sommerville (1879–1934)
[Sommerville **1**, pp. 51–52]

By 15.26, any two distinct lines are either intersecting, parallel, or ultraparallel. In other words, they belong to a *pencil* of lines of one of three kinds: an ordinary pencil, consisting of all the lines through one point, a pencil of parallels, consisting of all the lines parallel to a given ray, or a *pencil of ultraparallels,* consisting of all the lines perpendicular to a given line. By 15.32, the product of reflections in the two lines is a rotation, a parallel displacement, or a translation, respectively. By fixing one of the two lines and allowing the other to vary in the pencil, we see that each of these three kinds of direct isometry can be applied as a continuous motion.

A *circle* with center O may be defined either as in § 15.1 or to be the locus of a point P which is derived from a fixed point Q (distinct from O) by continuous rotation about O, or to be the locus of the image of Q by reflection in all the lines through O. When the radius OQ becomes infinite, we have a *horocycle* with center M (at infinity): the locus of a point which is derived from a fixed point Q by a continuous parallel displacement, or the locus of the image of Q by reflection in all the lines parallel to the ray QM [Coxeter **3**, p. 213]. The rays parallel to QM are called the *diameters* of the horocycle. The first "o" in the word "horocycle" is short, as in "horror."

The locus of a point at a constant distance from a fixed line o is not a pair of parallel lines, as it would be in the Euclidean plane, but an *equidistant curve* (or "hypercycle"), having two branches, one on each side of its *axis* o. Either branch may be described as the locus of a point which is derived from a fixed point Q (not on o) by continuous translation along o, or as the locus of the image of Q by reflection in all the lines perpendicular to o.

Orthogonal to the pencil of lines through O we have a pencil of concentric circles. A rotation about O permutes the lines and slides each circle along itself. Orthogonal to the pencil of parallels with a common end M we have a pencil of *concentric horocycles.* A parallel displacement with center M permutes the parallel lines and slides each horocycle along itself. Orthogonal to the pencil of ultraparallels perpendicular to o we have a pencil of *coaxal equidistant curves*. A translation along o permutes the ultraparallel lines and slides each equidistant curve along itself.

We are now ready to fulfill the promise, made after 15.31, to show that "the product of two translations with nonintersecting axes may be a rotation." Referring to Figure 16.3*d*, we see that the line through C perpendicular to AB is ultraparallel to AM and AN. Therefore, it has a common perpendicular GH with AM, and a common perpendicular FE with AN, forming a pentagon $AEFGH$ with right angles at E, F, G, H as in Figure 16.6*a*. The remaining angle (at A) may be as small as we please; if it is zero, the pentagon is "asymptotic." The product of reflections in AE and FG is a translation along EF (through $2EF$). The product of reflections in FG and AH is a translation along GH (through $2GH$). Hence the product of these two translations is the same as the product of reflections in AE and AH, which is a rotation or, if A is an "end," a parallel displacement. Since the axes of the two translations are both perpendicular to FG, we have thus proved that the product of translations along two ultraparallel lines may be either a rotation or a parallel displacement. (Of course, it may just as easily be another translation.)

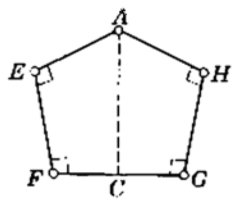


Figure 16.6a

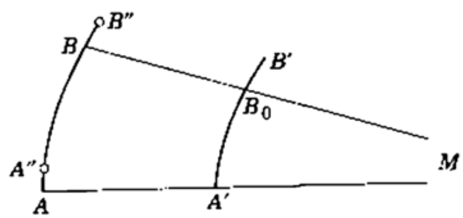


Figure 16.6b

The product of translations along two parallel lines, AM and BM, leaves invariant the common end M; therefore it cannot be a rotation, but must be either a translation along another line through M or a parallel displacement with center M. We shall soon see that the latter possibility arises when the two given translations are of equal length, one towards M and the other away from M. In fact, the translation along AM from A to A' (Figure 16.6*b*) transforms the arc AB of a horocycle through A into an equal arc $A'B'$ of the concentric horocycle through A'. Let B_0 denote the point in which the latter arc is cut by the diameter through B. The translation along this diameter from B_0 to B transforms the arc B_0A' of the second horocycle into the equal arc BA'' of the first. Thus the product of the two translations is the parallel displacement that transforms the arc AB into $A''B''$; it slides this horocycle (and every concentric horocycle) along itself.

EXERCISES

1. The three vertices of a (finite) triangle all lie on each of three equidistant curves, whose axes join midpoints of pairs of sides, and on a fourth "cycle," which may be either

a circle or a horocycle or another equidistant curve (with all three vertices on one branch). [Sommerville **1**, pp. 54, 189.]

2. The three sides of a (finite) triangle all touch a circle (the incircle) and three other "cycles," each of which may be of any one of the three kinds.

3. In Figure 15.2*a*, the horocycle through J with diameter p_1 passes also through L.

4. How many horocycles will pass through two given points?

5. An equidistant curve may have as many as four intersections with a circle or a horocycle or another equidistant curve.

6. Develop the analogy between conics in the affine plane and generalized circles in the hyperbolic plane. A horocycle, like a parabola, goes to infinity in one direction: if the points P and Q on it are variable and fixed, respectively, the limiting position of the line QP is the diameter through Q. An equidistant curve, like a hyperbola, has two branches.

7. Unlike the conjugate axis of a hyperbola, the axis of an equidistant curve is on the *concave* side of each branch.

16.7 POINCARÉ'S "HALF-PLANE" MODEL

There is a gain in simplicity when the fundamental circle is taken as a straight line, say the axis of x. . . . We may avoid dealing with pairs of points by considering only those points above the x-axis. A proper circle is represented by a circle lying entirely above the x-axis; a horocycle by a circle touching the x-axis; an equidistant-curve by the upper part of a circle cutting the x-axis together with the reflexion of the part which lies below the axis.

D. M. Y. Sommerville [**1**, pp. 188-189]

From the conformal model (Figure 16.2*a*) in which the lines are represented by circles (and lines) orthogonal to a fixed circle Ω, Poincaré derived another conformal model by inversion in a circle whose center lies on Ω. The inverse of Ω is a line, say a "horizontal" line, which we shall again denote by Ω. The points of the hyperbolic plane are represented by pairs of points which are images of each other by reflection in Ω, and the lines are represented by circles and lines orthogonal to Ω, that is, circles whose centers lie on Ω, and vertical lines [Burnside **1**, p. 387].

Through a pair of points which are images in Ω, we can draw an intersecting pencil of coaxal circles (like Figure 6.5*a* turned through a right angle) representing an ordinary pencil of lines. The orthogonal nonintersecting pencil, having Ω for its radical axis, represents a pencil of concentric circles. The limiting points of the nonintersecting pencil represent the common center of the concentric circles.

Another pencil of circles (situated as in Figure 6.5*a* itself) can be drawn through two points on Ω. One member of this pencil, having its center on Ω, represents a line o. The remaining circles (or strictly, pairs of them re-

lated by reflection in Ω) represent coaxal equidistant curves with axis o. For, the orthogonal nonintersecting pencil represents the pencil of ultraparallel lines perpendicular to o.

A tangent pencil of circles whose centers lie on Ω (Figure 6.5*b*) represents a pencil of parallels, whereas the orthogonal tangent pencil (touching Ω) represents a pencil of concentric horocycles. One particular pencil of parallels (special in the model but, of course, not special in the hyperbolic geometry itself) is represented by all the vertical lines (which pass, like Ω itself, through the point at infinity of the inversive plane). The horocycles having these lines for diameters are represented by all the horizontal lines except Ω (or strictly, pairs of such lines related by reflection in Ω). Since reflections in the vertical lines represent reflections in the parallel lines, horizontal translations represent parallel displacements. Hence the horizontal lines (other than Ω itself) represent the horocycles *isometrically:* equal segments represent equal arcs.

EXERCISES

1. What figure is represented by two lines forming an angle that is bisected by Ω?

2. When two ultraparallel lines are represented by nonintersecting circles (in either of Poincaré's conformal models), the distance between the lines, measured along their common perpendicular, appears as the *inversive distance* between the circles (see Exercise 5 of § 6.6).

3. The angle of parallelism (Figure 16.3*d* on page 293) is

$$\Pi(x) = 2 \arctan e^{-x}.$$

16.8 THE HOROSPHERE AND THE EUCLIDEAN PLANE

> *F. L. Wachter (1792-1817) . . . in a letter to Gauss (Dec., 1816) . . . speaks of the surface to which a sphere tends as its radius approaches infinity. . . . He affirms that even in the case of the Fifth Postulate being false, there would be a geometry on this surface identical with that of the ordinary plane.*
>
> R. Bonola [**1**, pp. 62-63]

The ideas in §§ 16.6 and 16.7 extend in an obvious manner from two to three dimensions. The locus of images of a point Q by reflection in all the planes through a point O is a *sphere* with radius OQ. As a limiting case we have a *horosphere* with center M (at infinity): the locus of images of a point Q by reflection in all the planes parallel to the ray QM [Coxeter **3**, p. 218]. The locus of images of a point Q by reflection in all the planes perpendicular to a fixed plane ω is one sheet of an *equidistant surface,* which consists of points at a constant distance from ω on either side.

There is a conformal model in inversive space in which the points of hyperbolic space are represented by pairs of points related by reflection in

a fixed "horizontal" plane Ω, and the planes are represented by spheres and planes orthogonal to Ω, that is, spheres whose centers lie on Ω, and vertical planes. The representation of lines (which are intersections of planes) follows immediately. Of particular interest is the bundle of vertical lines, which represents the bundle of lines parallel to a given ray QM (special in the model, though not in the hyperbolic geometry itself). The horospheres that have these lines for diameters are represented by all the horizontal planes except Ω. Since every vertical plane provides a model (of the kind described in § 16.7) for a plane in the hyperbolic space, each horizontal plane (except Ω) represents a horosphere, and every line in the plane represents a horocycle on the horosphere. Since distances along such lines agree with distances along the corresponding horocycles, the representation of the horosphere by the Euclidean plane is *isometric:* for any figure in the Euclidean plane there is a congruent figure on the horosphere (with lines replaced by horocycles).

This astonishing theorem was discovered independently by Bolyai and Lobachevsky. For two different proofs, see Coxeter [**3**, pp. 197, 251]. It means that, along with ordinary spherical geometry, the inhabitants of a hyperbolic world would also study horospherical geometry, which is the same as Euclidean geometry!

Part IV

17
Differential geometry of curves

Differential geometry is concerned with applying the methods of analysis to geometry, especially to the study of curves and surfaces. Classically, the study is made in Euclidean space of three dimensions. But in the twentieth century other spaces, such as inversive, affine, or projective, have been used. In other words, differential geometry is still significant when there is no concept of distance. However, both distance and parallelism are usually present, in which case the notion of a vector is fundamental.

A curve, being the locus of a point P, is intimately associated with a variable vector, namely the position vector

$$\mathbf{r} = \overrightarrow{OP}$$

which goes from a fixed origin O to the point P. For simplicity we shall consider only rectifiable curves for which there is a well-defined tangent at each point [Kreyszig **1**, p. 29].

After a preliminary discussion of vectors, we shall consider the curvature of plane curves, and the curvature and torsion of twisted curves, applying the results to many important special cases such as spirals and helices.

17.1 VECTORS IN EUCLIDEAN SPACE

We have already considered, in § 13.6, the *affine* properties of vectors, such as addition and subtraction, multiplication by numbers, independence, and the unique expression

17.11 $$\mathbf{c} = x\mathbf{e} + y\mathbf{f} + z\mathbf{g}$$

for any vector $\mathbf{c}$ as a linear combination of three basic vectors $\mathbf{e}$, $\mathbf{f}$, $\mathbf{g}$. The time has now come to introduce two kinds of multiplication of vectors by one another. We shall employ the notation of J. W. Gibbs (1839–1903), although some authors, such as Birkhoff and MacLane [**1**, p. 175] and Forder [**2**], prefer that of H. Grassmann (1809–1877).

Euclidean geometry allows us to speak of the *length* (or "magnitude," or "absolute value") $|\mathbf{a}|$, of any given vector $\mathbf{a}$. If θ is the angle between $\mathbf{a}$ and another vector $\mathbf{b}$, we define the *inner* (or "scalar") product $\mathbf{a} \cdot \mathbf{b}$ and the *outer* (or "vector") product $\mathbf{a} \times \mathbf{b}$ by the formulas

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos \theta, \qquad \mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \sin \theta\, \mathbf{g},$$

where $\mathbf{g}$ is the unit vector orthogonal to the plane $\mathbf{ab}$ on the side from which θ appears as a positive angle. The introduction of the auxiliary vector $\mathbf{g}$ (orthogonal to both $\mathbf{a}$ and $\mathbf{b}$) is justified by the elegant algebra that follows.

We see at once that, if m and n are numbers,

$$m\mathbf{a} \cdot n\mathbf{b} = mn\mathbf{a} \cdot \mathbf{b}, \qquad m\mathbf{a} \times n\mathbf{b} = mn\mathbf{a} \times \mathbf{b},$$

$$\mathbf{b} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{b}, \qquad \mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}.$$

Thus inner multiplication is commutative (like the multiplication of numbers), whereas outer multiplication is "anticommutative." Since $\mathbf{a} \times \mathbf{a} = \mathbf{0}$, we naturally take $\mathbf{a}^2$ to mean $\mathbf{a} \cdot \mathbf{a}$:

$$\mathbf{a}^2 = |\mathbf{a}|^2.$$

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal if $\mathbf{a} \cdot \mathbf{b} = 0$, parallel if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

Consider two vectors $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$, and let BN be the perpendicular from B to OA, as in Figure 17.1*a*. The algebraic distance ON (negative if $\angle AOB$ is obtuse) is called the *projection* of $\mathbf{b}$ on $\mathbf{a}$. If $|\mathbf{a}| = 1$, so that $\mathbf{a}$ is a *unit* vector, this projection is clearly $\mathbf{a} \cdot \mathbf{b}$. Removing the restriction to a unit vector, we find that $\mathbf{a} \cdot \mathbf{b}$ is $|\mathbf{a}|$ times the projection. This geometrical interpretation makes it easy to establish, for inner products, the distributive law

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{b}') = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{b}',$$

which may also be expressed as

$$(\mathbf{b} + \mathbf{b}') \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{a} + \mathbf{b}' \cdot \mathbf{a}$$

in virtue of the commutative law $\mathbf{b} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{b}$. Replacing $\mathbf{b}'$ by $-\mathbf{b}'$, we obtain the corresponding results for differences instead of sums.

The distributive law provides a useful method for establishing certain identities. If $\mathbf{b}$ and $\mathbf{b}'$ stand for expressions which we wish to prove equal, it is sometimes helpful to introduce an arbitrary vector $\mathbf{c}$ and to compare $\mathbf{b} \cdot \mathbf{c}$ with $\mathbf{b}' \cdot \mathbf{c}$. If we find that

$$\mathbf{b} \cdot \mathbf{c} = \mathbf{b}' \cdot \mathbf{c}$$

for all choices of $\mathbf{c}$ (or even for three independent $\mathbf{c}$'s), we can safely assert that $\mathbf{b} = \mathbf{b}'$. For, since $(\mathbf{b} - \mathbf{b}') \cdot \mathbf{c} = 0$, if $\mathbf{b} - \mathbf{b}'$ is not the zero vector it

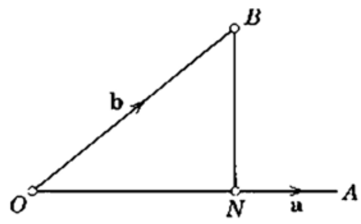


Figure 17.1a

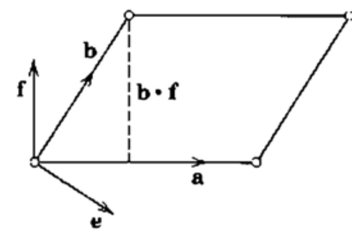


Figure 17.1b

must be orthogonal to $\mathbf{c}$; and, since $\mathbf{c}$ is arbitrary, this is impossible.

As a step towards establishing the distributive law for outer products, we compare two expressions for the area of a parallelogram, namely

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\ \mathbf{b}\cdot\mathbf{f},$$

where $\mathbf{f}$ is a unit vector orthogonal to $\mathbf{a}$ in the plane $\mathbf{ab}$ (as in Figure 17.1*b*) so that $\mathbf{b}\cdot\mathbf{f}$ is the altitude of the parallelogram from its base $|\mathbf{a}|$. Analogously, the parallelepiped formed by three independent vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ has base $|\mathbf{a} \times \mathbf{b}|$, altitude $\mathbf{c}\cdot\mathbf{g}$, and volume (with a suitable convention of sign, depending on whether the trihedron $\mathbf{abc}$ is positively or negatively oriented)

$$|\mathbf{a} \times \mathbf{b}|\ \mathbf{c}\cdot\mathbf{g} = |\mathbf{a} \times \mathbf{b}|\ \mathbf{g}\cdot\mathbf{c} = (\mathbf{a} \times \mathbf{b})\cdot\mathbf{c}.$$

Since we could just as well regard another face of the parallelepiped as its base, the same volume is expressible as

$$(\mathbf{b} \times \mathbf{c})\cdot\mathbf{a} = \mathbf{a}\cdot(\mathbf{b} \times \mathbf{c}).$$

Thus we can interchange the cross and the dot:

$$(\mathbf{a} \times \mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot(\mathbf{b} \times \mathbf{c}).$$

(This is as near as we can come to an "associative law" for products of vectors.) Since the dot and cross are interchangeable, it is convenient to use, for $(\mathbf{a} \times \mathbf{b})\cdot\mathbf{c}$ or $\mathbf{a}\cdot(\mathbf{b} \times \mathbf{c})$, the special symbol $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$, so that the volume of the parallelepiped is

$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = [\mathbf{b}\ \mathbf{c}\ \mathbf{a}] = [\mathbf{c}\ \mathbf{a}\ \mathbf{b}] = -[\mathbf{c}\ \mathbf{b}\ \mathbf{a}].$$

If $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$, the parallelepiped collapses, and the three vectors are coplanar, that is, dependent. Thus a necessary and sufficient condition for $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ to be independent is

$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \neq 0.$$

To prove the distributive law for outer products, we introduce an arbitrary vector $\mathbf{c}$ (like a catalyst) and find

$$\begin{aligned}\{(\mathbf{a} + \mathbf{a}') \times \mathbf{b}\}\cdot\mathbf{c} &= (\mathbf{a} + \mathbf{a}')\cdot(\mathbf{b} \times \mathbf{c})\\ &= \mathbf{a}\cdot(\mathbf{b} \times \mathbf{c}) + \mathbf{a}'\cdot(\mathbf{b} \times \mathbf{c})\\ &= (\mathbf{a} \times \mathbf{b})\cdot\mathbf{c} + (\mathbf{a}' \times \mathbf{b})\cdot\mathbf{c}\\ &= \{(\mathbf{a} \times \mathbf{b}) + (\mathbf{a}' \times \mathbf{b})\}\cdot\mathbf{c}.\end{aligned}$$

Since $\mathbf{c}$ is arbitrary, we conclude that

$$(\mathbf{a} + \mathbf{a}') \times \mathbf{b} = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a}' \times \mathbf{b}).$$

Since the outer product of two vectors is a vector, we might at first expect the associative law to hold. To see why $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ and $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ are, in general, different, let us evaluate both expressions, using a procedure devised by Coe and Rainich.* Consider unit vectors $\mathbf{e}$ and $\mathbf{f}$ in the plane $\mathbf{ab}$, orthogonal to $\mathbf{b}$ and $\mathbf{a}$ respectively, as in Figure 17.1b. Since the vector $\mathbf{a} \times \mathbf{b}$ is perpendicular to the plane $\mathbf{ab}$ (or $\mathbf{ef}$), the two vectors

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{e}, \qquad (\mathbf{a} \times \mathbf{b}) \times \mathbf{f}$$

lie in this plane and have the same length $|\mathbf{a} \times \mathbf{b}|$, which may be expressed in either of the forms

$$\mathbf{a}\cdot\mathbf{e}\,|\mathbf{b}|, \qquad \mathbf{b}\cdot\mathbf{f}\,|\mathbf{a}|.$$

Since they have the same directions as $\mathbf{b}$, $-\mathbf{a}$, respectively, they are exactly

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{e} = \mathbf{a}\cdot\mathbf{e}\,\mathbf{b}, \qquad (\mathbf{a} \times \mathbf{b}) \times \mathbf{f} = -\mathbf{b}\cdot\mathbf{f}\,\mathbf{a}.$$

If $\mathbf{g}$ is perpendicular to the plane, we have also

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{g} = \mathbf{0}.$$

Using the three vectors $\mathbf{e}$, $\mathbf{f}$, $\mathbf{g}$ as a basis, we may express an arbitrary vector $\mathbf{c}$ in the form 17.11; thus

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} &= (\mathbf{a} \times \mathbf{b}) \times (x\mathbf{e} + y\mathbf{f} + z\mathbf{g})\\ &= x(\mathbf{a} \times \mathbf{b}) \times \mathbf{e} + y(\mathbf{a} \times \mathbf{b}) \times \mathbf{f} + z(\mathbf{a} \times \mathbf{b}) \times \mathbf{g}\\ &= x(\mathbf{a}\cdot\mathbf{e})\,\mathbf{b} - y\,(\mathbf{b}\cdot\mathbf{f}\,)\,\mathbf{a} = (\mathbf{a}\cdot x\mathbf{e})\,\mathbf{b} - (\mathbf{b}\cdot y\mathbf{f}\,)\,\mathbf{a}\\ &= \mathbf{a}\cdot(x\mathbf{e} + y\mathbf{f} + z\mathbf{g})\,\mathbf{b} - \mathbf{b}\cdot(x\mathbf{e} + y\mathbf{f} + z\mathbf{g})\,\mathbf{a},\end{aligned}$$

since $\mathbf{a}\cdot\mathbf{f}$, $\mathbf{a}\cdot\mathbf{g}$, $\mathbf{b}\cdot\mathbf{e}$, $\mathbf{b}\cdot\mathbf{g}$ are all zero. Hence, finally,

17.12 $$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\,\mathbf{a}.$$

Interchanging $\mathbf{a}$ and $\mathbf{c}$, we deduce

$$\begin{aligned}\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \times \mathbf{b}) \times \mathbf{a} &= (\mathbf{c}\cdot\mathbf{a})\,\mathbf{b} - (\mathbf{b}\cdot\mathbf{a})\,\mathbf{c}\\ &= (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}.\end{aligned}$$

By considering $\{(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}\}\cdot\mathbf{d}$, we find also that any four vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ satisfy *Lagrange's identity*

17.13 $$(\mathbf{a} \times \mathbf{b})\cdot(\mathbf{c} \times \mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{d}).$$

* C. J. Coe and G. Y. Rainich, *American Mathematical Monthly*, **56** (1949), pp. 175–176.

It is sometimes desirable to express a vector in terms of its components in the directions of the axes of rectangular Cartesian coordinates, that is, to let the coordinate symbol (x, y, z) for the point P to be used also for the vector $\overrightarrow{OP}$, where O is the origin $(0, 0, 0)$. In other words, we use $P = (x, y, z)$ as an abbreviation for

17.14 $$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors along the three axes (so that this is a special case of 17.11). Since

17.15 $$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 1, \qquad \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = \mathbf{i}\cdot\mathbf{j} = 0,$$
$$\mathbf{i} = \mathbf{j} \times \mathbf{k}, \qquad \mathbf{j} = \mathbf{k} \times \mathbf{i}, \qquad \mathbf{k} = \mathbf{i} \times \mathbf{j}, \qquad [\mathbf{i}\,\mathbf{j}\,\mathbf{k}] = 1,$$

we easily deduce, for any three vectors $\mathbf{r}, \mathbf{r}', \mathbf{r}''$, the products

17.151 $$\mathbf{r}\cdot\mathbf{r}' = xx' + yy' + zz',$$
$$\mathbf{r} \times \mathbf{r}' = \begin{vmatrix} y & z \\ y' & z' \end{vmatrix}\mathbf{i} + \begin{vmatrix} z & x \\ z' & x' \end{vmatrix}\mathbf{j} + \begin{vmatrix} x & y \\ x' & y' \end{vmatrix}\mathbf{k}$$

and

17.16 $$[\mathbf{r}\,\mathbf{r}'\,\mathbf{r}''] = \begin{vmatrix} x & y & z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix}.$$

Since the product of two determinants (like the product of two matrices) is obtained by writing down the inner products of the rows of the first with the columns of the second, we can bring in three more vectors such as $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ and find

$$[\mathbf{q}\,\mathbf{q}'\,\mathbf{q}''][\mathbf{r}\,\mathbf{r}'\,\mathbf{r}''] = \begin{vmatrix} u & v & w \\ u' & v' & w' \\ u'' & v'' & w'' \end{vmatrix}\begin{vmatrix} x & x' & x'' \\ y & y' & y'' \\ z & z' & z'' \end{vmatrix}$$

17.17 $$= \begin{vmatrix} \mathbf{q}\cdot\mathbf{r} & \mathbf{q}\cdot\mathbf{r}' & \mathbf{q}\cdot\mathbf{r}'' \\ \mathbf{q}'\cdot\mathbf{r} & \mathbf{q}'\cdot\mathbf{r}' & \mathbf{q}'\cdot\mathbf{r}'' \\ \mathbf{q}''\cdot\mathbf{r} & \mathbf{q}''\cdot\mathbf{r}' & \mathbf{q}''\cdot\mathbf{r}'' \end{vmatrix}.$$

Returning to 17.14, we observe that

$$\mathbf{r}\cdot\mathbf{i} = x, \quad \mathbf{r}\cdot\mathbf{j} = y, \quad \mathbf{r}\cdot\mathbf{k} = z.$$

Thus we can express any vector $\mathbf{r}$ in terms of any orthogonal trihedron of unit vectors in the form

17.18 $$\mathbf{r} = (\mathbf{r}\cdot\mathbf{i})\,\mathbf{i} + (\mathbf{r}\cdot\mathbf{j})\,\mathbf{j} + (\mathbf{r}\cdot\mathbf{k})\,\mathbf{k}.$$

We shall also have occasion to use the following theorem:

17.19 *If two vectors,* $\mathbf{a}$ *and* $\mathbf{b}$*, lie in perpendicular planes which intersect the line of a unit vector* $\mathbf{k}$*, then*

$$(\mathbf{a}\cdot\mathbf{k})(\mathbf{k}\cdot\mathbf{b}) = \mathbf{a}\cdot\mathbf{b}.$$

Proof. Since the planes $\mathbf{ak}$ and $\mathbf{kb}$ are perpendicular, 17.13 yields

$$0 = (\mathbf{a}\times\mathbf{k})\cdot(\mathbf{k}\times\mathbf{b}) = (\mathbf{a}\cdot\mathbf{k})(\mathbf{k}\cdot\mathbf{b}) - \mathbf{k}^2(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\mathbf{k})(\mathbf{k}\cdot\mathbf{b}) - (\mathbf{a}\cdot\mathbf{b}).$$

EXERCISES

1. How must $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be related in order to satisfy the associative law $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \mathbf{a}\times(\mathbf{b}\times\mathbf{c})$?
2. Simplify $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$ two ways and, by equating the results, deduce an identity connecting four vectors such as $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,\mathbf{d}$.
3. Simplify $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b})$, and show that the result could have been anticipated in virtue of a well known trigonometrical identity.

17.2 VECTOR FUNCTIONS AND THEIR DERIVATIVES

Vector functions can be differentiated in the same manner as numerical functions. Let the vector

$$\mathbf{a} = \mathbf{a}(s)$$

be a function of the numerical variable s, and let $\Delta\mathbf{a}$ be the increment in the vector corresponding to the increment Δs in the variable s, so that

$$\mathbf{a}(s + \Delta s) = \mathbf{a} + \Delta\mathbf{a}.$$

If the vector $\Delta\mathbf{a}/\Delta s$ tends to a limit as Δs tends to zero, the vector function $\mathbf{a}(s)$ is said to be *differentiable*, and the limit is the *derivative:*

$$\dot{\mathbf{a}} = \frac{d\mathbf{a}}{ds} = \lim_{\Delta s\to 0}\frac{\Delta\mathbf{a}}{\Delta s} = \lim_{\Delta s\to 0}\frac{\mathbf{a}(s+\Delta s) - \mathbf{a}(s)}{\Delta s}.$$

The rule for differentiating a product is the same as for ordinary functions. In fact,

$$(\mathbf{a} + \Delta\mathbf{a})\cdot(\mathbf{b} + \Delta\mathbf{b}) - \mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\Delta\mathbf{b} + \Delta\mathbf{a}\cdot\mathbf{b} + \Delta\mathbf{a}\cdot\Delta\mathbf{b}$$

and therefore

$$\frac{d}{ds}(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot\dot{\mathbf{b}} + \dot{\mathbf{a}}\cdot\mathbf{b} - \lim \dot{\mathbf{a}}\cdot\dot{\mathbf{b}}\,\Delta s$$

$$= \mathbf{a}\cdot\dot{\mathbf{b}} + \dot{\mathbf{a}}\cdot\mathbf{b}.$$

Similarly,
$$\frac{d}{ds}(m\mathbf{a}) = m\dot{\mathbf{a}} + \dot{m}\mathbf{a}$$

and
$$\frac{d}{ds}(\mathbf{a}\times\mathbf{b}) = (\mathbf{a}\times\dot{\mathbf{b}}) + (\dot{\mathbf{a}}\times\mathbf{b}).$$

(Since outer products are anticommutative, we must be careful not to write the second term on the right as $\mathbf{b} \times \dot{\mathbf{a}}$.)

Since

$$\frac{d}{ds}\mathbf{a}^2 = 2\mathbf{a}\cdot\dot{\mathbf{a}},$$

a variable vector of constant length is always orthogonal to its derivative.

Since the Cartesian basic vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are constant, their derivatives are zero, and

$$\dot{\mathbf{r}} = \frac{d}{ds}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} = (\dot{x}, \dot{y}, \dot{z}).$$

Thus, when we differentiate a vector, the components of the derivative are simply the derivatives of the components.

EXERCISE

When a particle moves in a circular orbit (like a stone swung at the end of a string), its position vector from the center has constant length. In which direction is its velocity? If its speed is constant, its velocity is a vector of constant length. In which direction is its acceleration?

17.3 CURVATURE, EVOLUTES AND INVOLUTES

The simplest instance of a variable vector is the position vector $\mathbf{r} = \overrightarrow{OP}$ of a point P that moves along a curve (including a straight line as the simplest case of all). To define the length of an arc of the curve, we approximate it by a sequence of broken lines, as in §8.5. The increment $\Delta\mathbf{r}$ may be identified with the vector along one of the segments of the broken line, so that, before we pass to the limit, the corresponding increment of arc is the length $|\Delta\mathbf{r}|$.

For most purposes, the directed arc s (measured along the curve from a fixed point A to a variable point P) is the most convenient parameter to use in describing the curve. That is, we regard the vector $\mathbf{r} = \overrightarrow{OP}$ as a function of s. Since

$$\lim \frac{|\Delta\mathbf{r}|}{\Delta s} = \lim\left[\left(\frac{\Delta x}{\Delta s}\right)^2 + \left(\frac{\Delta y}{\Delta s}\right)^2 + \left(\frac{\Delta z}{\Delta s}\right)^2\right]^{\frac{1}{2}} = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{\frac{1}{2}} = 1$$

[Struik **1**, p. 7], the limit of $\Delta\mathbf{r}/\Delta s$ is the unit tangent vector

$$\mathbf{t} = \dot{\mathbf{r}}.$$

If another parameter u is used instead of s, we can easily make the necessary adjustment. The derivative $d\mathbf{r}/du$ is still a tangent vector, namely

$$\frac{d\mathbf{r}}{du} = \frac{d\mathbf{r}}{ds}\frac{ds}{du} = \frac{ds}{du}\mathbf{t};$$

the connection between s and u is determined by the length, ds/du, of this vector, and $\mathbf{t}$ is the unit vector in the same direction.

For instance, in the case of the circle

$$x = \rho \cos u, \qquad y = \rho \sin u,$$

of radius ρ, we have

$$\begin{aligned} \mathbf{r} &= \rho(\cos u, \sin u), \\ \frac{ds}{du}\mathbf{t} &= \rho(-\sin u, \cos u), \\ \frac{ds}{du} &= \rho, \qquad \mathbf{t} = (-\sin u, \cos u). \end{aligned}$$

For any curve in the (x, y)-plane, the tangent vector is

17.31 $$\mathbf{t} = (\cos \psi, \sin \psi),$$

where ψ is the angle that $\mathbf{t}$ makes with the vector $\mathbf{i}$ along the x-axis. The *curvature* of the plane curve is the arc derivative of this angle:

17.32 $$\kappa = \frac{d\psi}{ds} = \dot{\psi}.$$

Since $\mathbf{t}$ is a unit vector, its derivative is in the perpendicular direction, that is, in the direction of the unit *normal* vector $\mathbf{n} = (-\sin\psi, \cos\psi)$, which is derived from $\mathbf{t}$ by a positive quarter-turn. Thus

$$\dot{\mathbf{t}} = \dot{\psi}(-\sin\psi, \cos\psi)$$

17.33 $$= \kappa\mathbf{n},$$

and we regard κ as being positive or negative according as $\mathbf{n}$ is on the concave or convex side of the curve.

The derivative of $\mathbf{n}$, being orthogonal to $\mathbf{n}$, is a certain multiple of $\mathbf{t}$. By differentiating the inner product $\mathbf{n}\cdot\mathbf{t}$, which is zero, we find the precise expression

17.34 $$\dot{\mathbf{n}} = -\kappa\mathbf{t}.$$

Applying this method to the circle

$$\mathbf{r} = \rho(\cos u, \sin u),$$

for which $\mathbf{t} = (-\sin u, \cos u)$, we find

$$\kappa\mathbf{n} = \dot{\mathbf{t}} = \dot{u}(-\cos u, -\sin u),$$

whence

$$\kappa = \dot{u} = 1/\rho \quad \text{and} \quad \mathbf{n} = -(\cos u, \sin u).$$

This means that the curvature of the circle is the reciprocal of its radius, Soddy's "bend" (p. 15), and its normal is towards the center along the radius.

At a point P on any plane curve, the *center of curvature* P_c is the center of the *circle of curvature*, which is the circle of "closest fit," having the same normal and the same curvature. Its "radius"

17.35 $$\rho = 1/\kappa$$

(which we allow to be positive or negative with κ) is the *radius of curvature.* The center of curvature at P, being distant ρ from P along the normal there, has the position vector

17.36 $$\mathbf{r}_c = \mathbf{r} + \rho\mathbf{n}.$$

When P moves along the given curve (which we now assume not to be a circle nor a straight line), the center of curvature P_c moves along a related curve called the *evolute,** which may be expressed parametrically in terms of its own arc length s_c. Its unit tangent $\mathbf{t}_c$ is given by

$$\begin{aligned}\dot{s}_c\,\mathbf{t}_c &= \frac{d}{ds}(\mathbf{r} + \rho\mathbf{n}) = \dot{\mathbf{r}} + \rho\dot{\mathbf{n}} + \dot{\rho}\mathbf{n}\\ &= \mathbf{t} - \rho\kappa\mathbf{t} + \dot{\rho}\mathbf{n} = \dot{\rho}\mathbf{n}.\end{aligned}$$

Since $\mathbf{t}_c$ and $\mathbf{n}$ are unit vectors, it follows that

$$\dot{s}_c = \pm\dot{\rho} \quad \text{and} \quad \mathbf{t}_c = \pm\mathbf{n}:$$

the tangent at P_c to the evolute is the same line as the normal at P to the original curve (see Figure 17.3a). Thus the evolute, which we have defined as the locus of the center of curvature, could equally well be defined as the *envelope of normals.*

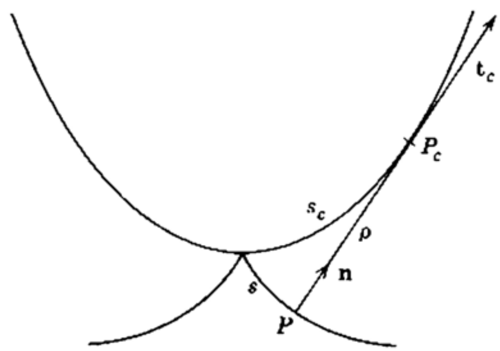


Figure 17.3a

Integrating the differential equation $ds_c = \pm d\rho$, we find that, for some constant a,

$$s_c = a \pm \rho.$$

Regarding the line PP_c as a rigid bar that rolls (without sliding) on the evolute, we now see that the end P of the bar traces out the original curve. In

* For a full discussion of this subject, see A. Ostrowski, Über die Evoluten von endlichen Ovalen, *Journal für die reine und angewandte Mathematik,* **198** (1957), pp. 14–27.

other words, the locus of P is an *involute* of the locus of P_c. We say "an involute" rather than "the involute" because different choices of the tracing point on the rolling bar yield an infinite family of "parallel" curves, each of which is an involute.

By a change of notation (from $\mathbf{r}_c$, s_c, $\mathbf{t}_c$ to $\mathbf{r}$, s, $\mathbf{t}$) we can assert that the position vector of a point which traces out an involute of a given curve is

$$\mathbf{r} + (a - s)\mathbf{t}.$$

To find $\mathbf{t}$, $\mathbf{n}$, and κ for a particular curve, the procedure that we applied to a circle (just after 17.34) is usually effective whenever the Cartesian coordinates are given in terms of a parameter. However, in the case of a central conic, the best way to obtain the evolute is as the envelope of normals. (See Ex. 3 at the end of § 8.5.)

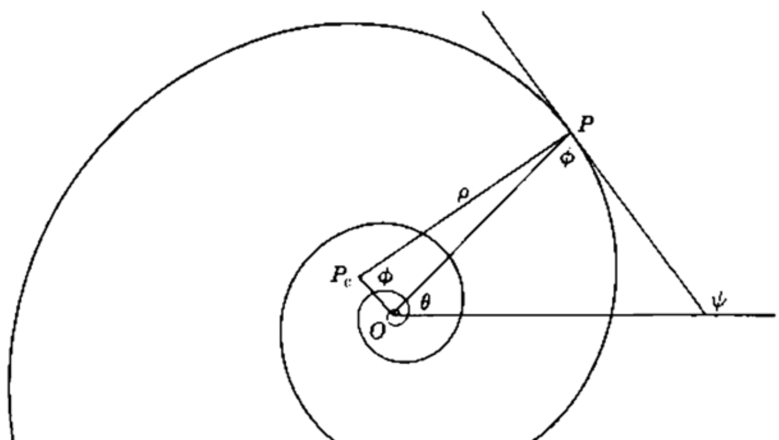


Figure 17.3b

For curves given in terms of polar coordinates, a more geometrical procedure may be desirable. For instance, to locate the center of curvature P_c at any point P on the equiangular spiral 8.71, we observe (Figure 17.3*b*) that $\psi = \theta + \phi$. Since also

$$\frac{dr}{ds} = \cos\phi \quad \text{and} \quad \frac{dr}{d\theta} = r\cot\phi,$$

we have

$$\kappa = \frac{d\psi}{ds} = \frac{d\theta}{ds} = \frac{d\theta}{dr}\frac{dr}{ds} = \frac{\sin\phi}{r},$$

so that

$$PP_c = \rho = r\csc\phi.$$

Thus OP_c is orthogonal to OP [Lamb **2**, p. 337] and P_c is (r_c, θ_c) where

$$r_c = r\cot\phi, \quad \theta_c = \theta + \tfrac{1}{2}\pi.$$

Since $r = r_c \tan \phi$ and $\theta = \theta_c - \frac{1}{2}\pi$, the evolute has the equation

$$r \tan \phi = a\mu^{\theta - \frac{1}{2}\pi}.$$

Since

$$\tan \phi = \mu^{\log \tan \phi / \log \mu} = \mu^{\log \tan \phi / \cot \phi} = \mu^{\tan \phi \log \tan \phi},$$

this is equivalent to

$$r = a\mu^{\theta - \frac{1}{2}\pi - \tan \phi \log \tan \phi},$$

which shows that the evolute is derived from the original spiral by a suitable rotation. (This result could have been seen from simple geometric principles, since the dilative rotation that slides the original spiral along itself must also slide the evolute along itself.)

The spiral is *its own evolute* if the "suitable rotation" consists of n whole turns, that is, if there is a positive integer n for which

$$\tfrac{1}{2}\pi + \tan \phi \log \tan \phi = 2n\pi.$$

This happens if $\tan \phi$ satisfies the transcendental equation

$$x \log x = (2n - \tfrac{1}{2})\pi.$$

From a table of natural logarithms we see that there is a unique solution for each positive integer n. The values $n = 1$ and $n = 2$ yield $\phi = 74° \, 39'$ [Cundy and Rollett **1**, p. 64] and $\phi = 80° \, 41'$. When n increases, ϕ approaches 90° and the spiral acquires a smaller and smaller "pitch."

EXERCISES

1. Find the evolute of the cycloid

$$x = u + \sin u, \qquad y = 1 + \cos u.$$

(*Hint:* $\mathbf{t} = (\cos \frac{1}{2}u, \, -\sin \frac{1}{2}u)$. A synthetic treatment is given by Lamb [**2**, pp. 351–352].)

2. Find the involute of the circle

$$x = \cos u, \qquad y = \sin u,$$

beginning at the point where $u = 0$.

3. From "simple geometric principles," the radius of curvature of an equiangular spiral is proportional to the arc s, measured from the origin. In fact,

$$\rho = s \cot \phi.$$

17.4 THE CATENARY

The *catenary* is an infinite curve, the idealized shape of a uniform chain hanging freely under the action of gravity. The curve evidently lies in a plane, which we may take to be the (x, y)-plane with the y-axis vertical, as in Figure 17.4*a*. Let W denote the weight of a unit length of the chain. We

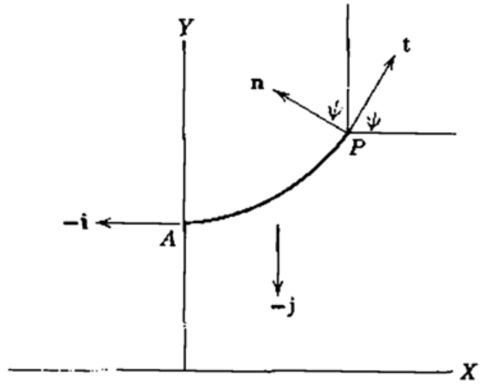


Figure 17.4a

consider the forces acting on arc AP, where A is the lowest point ($s = 0$) and P is at distance s, measured along the curve. The tangent $\mathbf{t}$ at P makes a certain angle ψ with the x-axis $\mathbf{i}$, and the normal $\mathbf{n}$ makes the same angle with the y-axis $\mathbf{j}$, so that

$$\mathbf{i}\cdot\mathbf{t} = \mathbf{j}\cdot\mathbf{n} = \cos\psi, \quad \mathbf{i}\cdot\mathbf{n} = -\sin\psi.$$

By considering various points P on the same chain, we may regard the inclination ψ as a function of the arc s, or vice versa, while conditions at A remain constant. The three forces that act on the arc AP are: the tension T at P, acting along the tangent $\mathbf{t}$, the tension Wa (equivalent to the weight of a certain length a of the chain) along the tangent $-\mathbf{i}$ at A, and the weight Ws in the direction $-\mathbf{j}$. Since these three forces are in equilibrium, we have

$$T\mathbf{t} - Wa\mathbf{i} - Ws\mathbf{j} = \mathbf{0}.$$

To eliminate the unknown (and uninteresting) tension T, we take the inner product with $\mathbf{n}$, obtaining

$$Wa\sin\psi - Ws\cos\psi = 0,$$

whence

17.41 $$s = a\tan\psi.$$

This equation, expressing the arc-length as a function of the inclination ψ, is called the *intrinsic* equation of the catenary. To deduce the Cartesian equation [cf. Lamb **2**, p. 290] we observe that

$$dx = ds\cos\psi, \qquad dy = ds\sin\psi$$

(Figure 8.5a) and make the "Gudermannian substitution"

$$\cosh u = \sec\psi, \qquad \sinh u = \tan\psi$$

(Figure 17.4b), which implies

$$\sinh u\,du = \sec\psi\tan\psi\,d\psi,$$
$$du = \sec\psi\,d\psi.$$

Differentiating 17.41, we obtain $ds = a \sec^2 \psi \, d\psi$, whence

$$dx = ds \cos \psi = a \sec \psi \, d\psi = a \, du,$$
$$dy = ds \sin \psi = a \sec \psi \tan \psi \, d\psi = a \, d(\sec \psi) = a \, d(\cosh u).$$

Taking the lowest point A (where $s = 0$, $\psi = 0$, and $u = 0$) to be $(0, a)$, we deduce

$$x = au, \qquad y = a \cosh u$$

or, in a single equation,

17.42
$$y = a \cosh \frac{x}{a}.$$

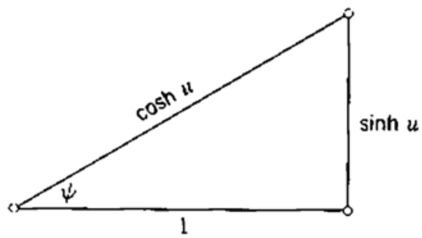


Figure 17.4b

EXERCISES

1. A uniform chain OP is held at P and hangs over a smooth peg at A, so placed that the chain just above A is horizontal and the peg gives it a right-angled bend. If the part of the chain from A to P is in the position indicated in Figure 17.4a, where is the free end O?

2. For the catenary, $s = a \sinh u$ and $\rho = a \cosh^2 u$.

3. Deduce 17.42 from $\dfrac{ds}{dx} = \sec \psi = \left[1 + \left(\dfrac{s}{a}\right)^2\right]^{\frac{1}{2}}$ and $\dfrac{dy}{dx} = \dfrac{s}{a}$.

4. Obtain intrinsic equations for (a) the cycloid $x = u + \sin u$, $y = \cos u$; (b) the parabola $y^2 = 2lx$.

5. Use the Gudermannian substitution to work out $\int \sec \psi \, d\psi$.

17.5 THE TRACTRIX

Let us now investigate the involute of the catenary, unwinding from its "lowest" point A, as in Figure 17.5a [Steinhaus **2**, pp. 212–213]. Since the position vector of the general point P on the catenary is

$$\mathbf{r} = (au, a \cosh u) = a(u, \cosh u),$$

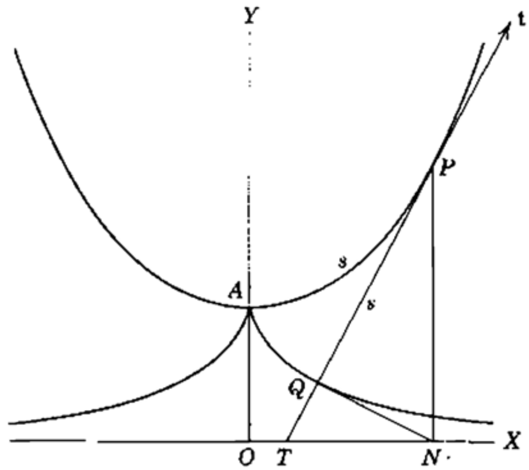


Figure 17.5a

where u is given in terms of s by the relation $s = a \sinh u$, the unit tangent vector is given by

$$a \cosh u \, \mathbf{t} = \frac{ds}{du}\frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{du} = a(1,\ \sinh u),$$
$$\mathbf{t} = (\operatorname{sech} u,\ \tanh u),$$

and the position vector of the general point Q on the involute is

$$\begin{aligned}\mathbf{r} - s\mathbf{t} &= a(u, \cosh u) - a \sinh u\,(\operatorname{sech} u, \tanh u)\\ &= a(u - \tanh u,\ \operatorname{sech} u).\end{aligned}$$

Thus the involute, which is known as the *tractrix,* has the parametric equations

17.51 $$x = a(u - \tanh u), \qquad y = a \operatorname{sech} u$$

[Lamb **2,** p. 325], from which there is no advantage in trying to eliminate u.

Since the unit normal vector at P to the catenary is $(-\tanh u, \operatorname{sech} u)$, the unit tangent vector at Q to the tractrix is $(\tanh u, -\operatorname{sech} u)$, and the position vector of the point N at distance a along it is

$$a(u - \tanh u, \operatorname{sech} u) + a(\tanh u, -\operatorname{sech} u) = (au, 0).$$

Thus the length of this tangent QN, from its point of contact to its intersection with the x-axis, has the constant value a. This is the property that gives the tractrix its name: if the (x, y)-plane is horizontal and you walk along the x-axis dragging a stone (originally at A) by means of a string of length a, the path of the stone is the tractrix. The x-axis is clearly an asymptote.

Another way of expressing the same property is that the tractrix is an orthogonal trajectory of a system of congruent circles whose centers lie on a

straight line. E. H. Lockwood* has developed this idea into an approximate construction for both the tractrix and the catenary.

EXERCISE

Compute ρ for the tractrix. What is its value at the "cusp" A, where $u = 0$?

17.6 TWISTED CURVES

We saw, in 7.52, that every displacement is a rotation or a translation or a twist (that is, the product of a rotation and a translation). As G. Mozzi remarked in 1763, this description evidently holds not only for a finite displacement but also for a continuous displacement: in the most general motion of a rigid body, there is at each instant a definite screw axis. In the case of a pure rotation, or of the motion of a screw in its nut, this axis remains invariant; but in general it is continually changing. For instance, the instantaneous axis of a wheel rolling along a road is not the line of the axle (which is moving as fast as the vehicle) but a parallel line on the road.

Any rotation may be described by its effect on a variable orthogonal trihedron of unit vectors which, for reasons that will appear a little later, we denote by **tpb,** so that

17.61
$$\mathbf{t}^2 = \mathbf{p}^2 = \mathbf{b}^2 = 1, \qquad \mathbf{p}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{t} = \mathbf{t}\cdot\mathbf{p} = 0,$$
$$\mathbf{t} = \mathbf{p}\times\mathbf{b}, \quad \mathbf{p} = \mathbf{b}\times\mathbf{t}, \quad \mathbf{b} = \mathbf{t}\times\mathbf{p}, \quad [\mathbf{t}\,\mathbf{p}\,\mathbf{b}] = 1$$

(cf. 17.15). We regard these unit vectors as functions of a parameter s. Since the derivative of any unit vector is in a perpendicular direction, the derivative of each of $\mathbf{t}$, $\mathbf{p}$, $\mathbf{b}$ lies in the plane of the other two and is a linear combination of them. Differentiating the relation $\mathbf{p}\cdot\mathbf{b} = 0$, we see that the coefficient of $\mathbf{p}$ in the expression for $\dot{\mathbf{b}}$ differs only in sign from the coefficient of $\mathbf{b}$ in the expression for $\dot{\mathbf{p}}$; similarly for the other pairs of vectors. Hence, for suitable numbers κ, λ, τ (functions of s), we have

17.62
$$\dot{\mathbf{t}} = \kappa\mathbf{p} - \lambda\mathbf{b}, \quad \dot{\mathbf{p}} = \tau\mathbf{b} - \kappa\mathbf{t}, \quad \dot{\mathbf{b}} = \lambda\mathbf{t} - \tau\mathbf{p}.$$

These derivatives are conveniently expressed in terms of *Darboux's vector*

$$\mathbf{d} = \tau\mathbf{t} + \lambda\mathbf{p} + \kappa\mathbf{b}.$$

For, we easily verify that

$$\dot{\mathbf{a}} = \mathbf{d}\times\mathbf{a},$$

where $\mathbf{a} = \mathbf{t}$ or $\mathbf{p}$ or $\mathbf{b}$ or any other vector rigidly attached to the moving trihedron [cf. Kreyszig **1,** p. 44]. We may even omit the variable vector $\mathbf{a}$ and write "symbolically"

* *Mathematical Gazette,* **43** (1959), pp. 117–118.

17.63 $$\frac{d}{ds} = \mathbf{d}\times.$$

At any point P on a twisted curve, the unit tangent vector $\mathbf{t} = \dot{\mathbf{r}}$ can be defined in the same way as for a plane curve. But instead of a unique normal orthogonal to $\mathbf{t}$, we have a *normal plane* containing a whole flat pencil of normals. Among the unit normal vectors, we give special names to two: the *principal normal* $\mathbf{p}$, in the direction of $\dot{\mathbf{t}}$, and the *binormal*

$$\mathbf{b} = \mathbf{t}\times\mathbf{p},$$

perpendicular to the plane $\mathbf{tp}$. Since this plane contains the derivative of $\mathbf{t}$ as well as $\mathbf{t}$ itself, its order of contact with the curve is higher than that of any other plane through $\mathbf{t}$. Because of the more intimate contact, we call $\mathbf{tp}$ the *osculating plane* at the point P. (It contains the directions of velocity and acceleration of a point moving along the curve [Forder **3**, p. 131].)

The formulas 17.62 for the derivatives of $\mathbf{t}$, $\mathbf{p}$, $\mathbf{b}$ are applicable, with the simplification $\lambda = 0$ due to our choice of $\mathbf{p}$ in the direction of $\dot{\mathbf{t}}$. Thus we have the *Serret-Frenet formulas*

17.64 $$\begin{aligned} \dot{\mathbf{t}} &= \kappa\mathbf{p}, \\ \dot{\mathbf{p}} &= \tau\mathbf{b} - \kappa\mathbf{t}, \\ \dot{\mathbf{b}} &= \quad\;\; - \tau\mathbf{p}, \end{aligned}$$

which may be epitomized in the form 17.63 with

17.65 $$\mathbf{d} = \kappa\mathbf{b} + \tau\mathbf{t}.$$

The coefficients κ and τ are called the *curvature* and *torsion* of the curve (at P).

When κ is constantly zero, $\mathbf{t}$ never changes and the "curve" is a straight line. As the name "curvature" suggests, κ measures the rate at which any nonstraight curve tends to depart from its tangent. Like a plane curve, a twisted curve has a *circle of curvature* of radius $1/\kappa$, which lies in the osculating plane and has its center on the principal normal; that is, the position vector of its center is $\mathbf{r} + \rho\mathbf{p}$, where $\rho = 1/\kappa$ is the radius of curvature.

When τ is constantly zero, the osculating plane never changes, and we have a *plane* curve, with $\mathbf{n} = \mathbf{p}$. The torsion (so named by L. I. Vallée in 1825) measures the rate at which a twisted curve tends to depart from its osculating plane.

The formulas 17.64 were first given by Serret (1851) and Frenet (1852) without the vector notation, that is, as formulas for the derivatives of the direction cosines of the tangent, principal normal, and binormal. Combining them with

$$\dot{\mathbf{r}} = \mathbf{t},$$

we obtain $$\ddot{\mathbf{r}} = \kappa\mathbf{p}, \qquad \dddot{\mathbf{r}} = \dot{\kappa}\mathbf{p} + \kappa(\tau\mathbf{b} - \kappa\mathbf{t}),$$

whence $$|\ddot{\mathbf{r}}| = \kappa, \qquad [\dot{\mathbf{r}}\;\ddot{\mathbf{r}}\;\dddot{\mathbf{r}}] = \kappa^2\tau.$$

EXERCISES

1. For a curve drawn on a sphere, the center of the circle of curvature at any point is the foot of the perpendicular from the center of the sphere upon the osculating plane at the point.

2. The tangent to the locus of the center of the circle of curvature of any curve is perpendicular to the tangent at the corresponding point on the original curve.

3. For any twisted curve, $[\dot{\mathbf{t}}\ \ddot{\mathbf{t}}\ \dddot{\mathbf{t}}] = \kappa^3(\kappa\dot{\tau} - \dot{\kappa}\tau) = \kappa^5 \dfrac{d}{ds}\left(\dfrac{\tau}{\kappa}\right),$

$$[\dot{\mathbf{b}}\ \ddot{\mathbf{b}}\ \dddot{\mathbf{b}}] = \tau^3(\dot{\kappa}\tau - \kappa\dot{\tau}) = \tau^5 \frac{d}{ds}\left(\frac{\kappa}{\tau}\right).$$

17.7 THE CIRCULAR HELIX

As we saw in § 8.7, the locus of a point moving in a plane under the action of a continuous dilative rotation is an equiangular spiral. Analogously, the locus of a point moving in space under the action of a continuous twist is a *circular helix* (§ 11.5). In terms of *cylindrical* coordinates (r, θ, z), defined by

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z \text{ as usual,}$$

a twist along and around the z-axis is

$$(r,\ \theta,\ z) \to (r,\quad \theta + u,\quad z + uc):$$

the product of the rotation $\theta \to \theta + u$ and the translation $z \to z + uc$. Applying this twist to the point $(a, 0, 0)$, we obtain (a, u, uc). Thus the circular helix has the parametric equations

$$r = a, \qquad \theta = u, \qquad z = uc,$$

or

$$x = a\cos u, \quad y = a\sin u, \quad z = cu$$

[Weatherburn **2**, p. 16]. In other words, the helix, which is the shape of the rail of a "spiral" staircase, has the equations

$$r = a, \qquad z = c\theta$$

or

$$x^2 + y^2 = a^2, \qquad \frac{y}{x} = \tan\frac{z}{c},$$

which express it as the curve of intersection of two surfaces: the circular cylinder

$$r = a, \quad \text{or} \quad x^2 + y^2 = a^2,$$

and the *helicoid*

$$z = c\theta, \quad \text{or} \quad \frac{y}{x} = \tan\frac{z}{c},$$

which is the shape of the ceiling of the staircase (or of a propellor blade) [Steinhaus **2**, p. 196].

Differentiating

$$\mathbf{r} = (a \cos u,\ a \sin u,\ cu)$$

with respect to s, we obtain

$$\mathbf{t} = \dot{u}\,(-a \sin u,\ a \cos u,\ c).$$

Since this must be a *unit* vector, we have

$$\dot{u} = 1/\sqrt{a^2 + c^2},$$

and we shall find it convenient to retain the symbol $\dot{u}$ as a temporary abbreviation for this constant. The Serret-Frenet formulas yield

$$\begin{aligned}
\kappa\mathbf{p} &= \dot{\mathbf{t}} = \dot{u}^2(-a\cos u,\ -a\sin u,\ 0) = -\dot{u}^2 a\,(\cos u,\ \sin u,\ 0),\\
\kappa &= \dot{u}^2 a = \frac{a}{a^2 + c^2},\\
\mathbf{p} &= -(\cos u,\ \sin u,\ 0) \qquad\qquad \text{(perpendicular to the } z\text{-axis)},\\
\mathbf{b} &= \mathbf{t} \times \mathbf{p} = \dot{u}\,(c \sin u,\ -c\cos u,\ a),\\
-\tau\mathbf{p} &= \dot{\mathbf{b}} = \dot{u}^2 c\,(\cos u,\ \sin u,\ 0),\\
\tau &= \dot{u}^2 c = \frac{c}{a^2 + c^2}.
\end{aligned}$$

Thus both κ and τ are constant, a result which could have been seen from first principles without any appeal to calculus, since the twist that slides the circular helix along itself transforms the curvature and torsion at one point into the same properties at another point. Conversely, since every displacement is a twist, every curve whose curvature and torsion are constant is a circular helix if we include, as limiting cases, the straight line ($\kappa = 0$, $a = 0$) and the circle ($\tau = 0$, $c = 0$).

When κ and τ are constant, Darboux's vector 17.65, being rigidly attached to the moving trihedron, is one of the vectors to which 17.63 is applicable. Thus

$$\dot{\mathbf{d}} = \mathbf{d} \times \mathbf{d} = \mathbf{0},$$

and $\mathbf{d}$ is constant. In fact, just as the motion of the tangent at a point describing a plane curve is, at each instant, a rotation about the center of curvature, so the motion of the **tpb** trihedron at a point describing a twisted curve is, at each instant, a twist about a certain line (*screw axis*) in the direction of Darboux's vector. In the case of the plane curve, the center of curvature appeared as the center of the circle of "closest fit": having the same tangent and curvature as the given curve. Analogously for the twisted curve, the screw axis can be obtained as the axis of the circular helix of closest fit: having the same **tpb** and the same curvature and torsion. Thus the screw axis is the line in the direction $\kappa\mathbf{b} + \tau\mathbf{t}$ through the point whose position vector is

$$\mathbf{r} + a\mathbf{p},$$

where a, being the radius of the circular cylinder containing the helix, is obtained by eliminating c from the equations

$$\kappa = \frac{a}{a^2 + c^2}, \qquad \tau = \frac{c}{a^2 + c^2};$$

in fact,

$$a = \frac{\kappa}{\kappa^2 + \tau^2}.$$

In the case of a plane curve we have $\tau = 0$, $a = \rho$, the position vector $\mathbf{r} + a\mathbf{p}$ becomes $\mathbf{r} + \rho\mathbf{n}$, and Darboux's vector becomes $\kappa\mathbf{b}$: perpendicular to the plane of the curve.

EXERCISES

1. The orthogonal projection of the circular helix on a plane through its axis, such as the plane $x = 0$, is the *sine curve*

$$y = a \sin \frac{z}{c}.$$

2. Describe the surface formed by the midpoints of all the chords of a circular helix.

3. The locus of centers of circles of curvature of a circular helix H is another circular helix H', and the locus of centers of circles of curvature of H' is H itself. For what value of c/a (or τ/κ) will H and H' be congruent? (It is, of course, sufficient to consider a single point on H, such as the point where $u = 0$.)

17.8 THE GENERAL HELIX

We have seen that the circular helix is characterized by its property of having constant curvature and constant torsion. It is a special case of the general *helix*, which may be defined either as a curve whose curvature and torsion are in a constant ratio or as a curve whose tangent makes a constant angle with a fixed vector. We proceed to prove the equivalence of these two definitions.

Suppose first that the curvature and torsion are in a constant ratio (i.e., a ratio independent of s), say

$$\tau = c\kappa.$$

Then

$$\dot{\mathbf{t}} = \kappa\mathbf{p}, \qquad \dot{\mathbf{b}} = -\tau\mathbf{p} = -c\kappa\mathbf{p},$$
$$c\dot{\mathbf{t}} + \dot{\mathbf{b}} = \mathbf{0}.$$

Since this is the derivative of $c\mathbf{t} + \mathbf{b}$, the latter is a fixed vector, say $\boldsymbol{\alpha}$, which makes a constant angle with $\mathbf{t}$ since

$$\boldsymbol{\alpha} \cdot \mathbf{t} = (c\mathbf{t} + \mathbf{b}) \cdot \mathbf{t} = c.$$

Conversely, suppose $\mathbf{t}$ makes a constant angle β with a fixed unit vector $\mathbf{k}$. Differentiating the equation

$$\mathbf{t}\cdot\mathbf{k} = \cos\beta,$$

we obtain
$$\kappa\mathbf{p}\cdot\mathbf{k} = 0.$$

Assuming that $\kappa \neq 0$, we have $\mathbf{p}\cdot\mathbf{k} = 0$, so that the constant vector $\mathbf{k}$ lies in the $\mathbf{bt}$ plane and makes complementary angles with $\mathbf{b}$ and $\mathbf{t}$. Since $\mathbf{t}\cdot\mathbf{k} = \cos\beta$, we have also

$$\mathbf{b}\cdot\mathbf{k} = \sin\beta.$$

Differentiating the equation $\mathbf{p}\cdot\mathbf{k} = 0$, we obtain

$$(\tau\mathbf{b} - \kappa\mathbf{t})\cdot\mathbf{k} = 0,$$

$$\tau\sin\beta - \kappa\cos\beta = 0,$$

$$\frac{\kappa}{\tau} = \tan\beta.$$

Lines in the constant direction $\mathbf{k}$ through all the points of the curve generate a (general) cylinder. Thus the helix can alternatively be described as a curve drawn on a cylinder in such a way as to cut the generators at a constant angle. In other words, it can be obtained by drawing a straight line obliquely on a sheet of paper and then wrapping the paper on the cylinder.

EXERCISES

1. Using Darboux's operator 17.63 to differentiate the constant vector $\mathbf{k}$, obtain

$$\mathbf{d}\times\mathbf{k} = \mathbf{0}.$$

Deduce that Darboux's vector $\mathbf{d} = \kappa\mathbf{b} + \tau\mathbf{t}$ is parallel to $\mathbf{k}$: its direction is constant (though its length, $\sqrt{\kappa^2 + \tau^2}$, may vary).

2. Find κ and τ for the curve

$$x = 3u - u^3, \quad y = 3u^2, \quad z = 3u + u^3,$$

and deduce that this curve is a helix.

17.9 THE CONCHO-SPIRAL

The spirals described on shells, and called concho-spirals, are such as would result from winding plane logarithmic spirals on cones.

Henry Moseley (1801-1872)

[*Moseley* **1**, *p.* 301]

The two most interesting helices are as follows: (1) the circular helix, which is the locus of a point under the action of a continuous twist, so that its curvature and torsion are constant; and (2) the *concho-spiral*, which is the locus of a point under the action of a continuous dilative rotation, so that its curvature and torsion are both inversely proportional to its arc s, measured from the apex O of the cone on which it evidently lies (cutting

the generators at a constant angle). A considerable arc of this curve can be seen on the shell *Turritella duplicata* [Weyl **1**, p. 68]. Architectural applications appear on spires in Copenhagen, notably that of the Stock Exchange Building, where the tails of four dragons are twisted together.

In terms of cylindrical coordinates, a dilative rotation round the z-axis, say

$$(r, \theta, z) \to (\mu^u r, \quad \theta + u, \quad \mu^u z),$$

applied to the point $(a, 0, c)$, yields the concho-spiral

$$r = \mu^u a, \quad \theta = u, \quad z = \mu^u c.$$

To see how the circular helix can arise as a limiting form of the concho-spiral, we change the origin by writing $z + c$ for z, and then make c tend to infinity and μ to 1 in such a manner that $(\mu - 1)c$ approaches a finite number b. Instead of $r = \mu^u a$ and $z = \mu^u c$, we have $r = a$ and

$$z = (\mu^u - 1)\, c = \frac{\mu^u - 1}{\mu - 1}\,(\mu - 1)c \to ub.$$

Thus the limiting form is the circular helix

$$r = a, \quad \theta = u, \quad z = ub.$$

EXERCISES

1. Express the parametric equations for the concho-spiral in terms of Cartesian coordinates.

2. Verify from these equations that the tangent $\mathbf{t}$ to the concho-spiral makes a constant angle with the z-axis.

3. Obtain a formula for the angle at which the concho-spiral cuts the generators of the cone

$$\frac{r}{a} = \frac{z}{c}.$$

4. A familiar model for a cone of revolution is obtained by cutting out a circular sector from a sheet of paper and rolling it up so that the center of the circle becomes the vertex of the cone. The angle α of the sector and the semivertical angle β of the cone are connected by the formula

$$\alpha = 2\pi \sin \beta;$$

for example, the sector is a semicircle if $\beta = \pi/6$. If $\sin \beta = 1/n$, where n is an integer greater than 1, the unfolded form of any concho-spiral on the cone consists of a sequence of arcs belonging to n equiangular spirals.

5. Like any other helix, the concho-spiral lies on a cylinder and cuts the generators at a constant angle. What kind of cylinder is this in the present case?

18

The tensor notation

In this interlude between the differential geometry of curves (Chapter 17) and the differential geometry of surfaces (Chapter 19) we introduce Ricci's famous notation, which is both suggestive and economical. (Without its aid the general theory of relativity could hardly have been formulated.) One of its simplest applications has no direct connection with differential geometry: "reciprocal lattices" are used in both x-ray crystallography (§ 18.3) and the geometry of numbers (§ 18.4).

18.1 DUAL BASES

The tensor method . . . has the great advantage that it is not a new notation, but a concise way of writing the ordinary notation.

Harold Jeffreys (1891 -)

[Jeffreys **1**, Preface]

As a basis for our vector space (or a frame for affine coordinates), instead of **e**, **f**, **g** (as in 17.11) or **i**, **j**, **k** (as in 17.14), it is more systematic to write $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$. Along with this set of three independent vectors we use also the *dual basis* $\mathbf{r}^1, \mathbf{r}^2, \mathbf{r}^3$, defined in terms of $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ by the equation

18.11 $$\mathbf{r}^\alpha \cdot \mathbf{r}_\beta = \delta^\alpha_\beta,$$

where the *Kronecker delta* δ^α_β is a useful symbol which means 1 or 0 according as α and β are equal or unequal. (The "1, 2, 3" of the dual basis are not exponents but "upper indices" or "superscripts," analogous to the subscripts used in the original basis.) Thus $\mathbf{r}^1$ is perpendicular to the plane $\mathbf{r}_2\mathbf{r}_3$ and its length is adjusted so that $\mathbf{r}^1 \cdot \mathbf{r}_1 = 1$; similarly for $\mathbf{r}^2$ and $\mathbf{r}^3$. Each $\mathbf{r}^\alpha$, being perpendicular to two $\mathbf{r}_\beta$'s, may be expressed as an outer product:

18.12 $$\mathbf{r}^1 = \frac{\mathbf{r}_2 \times \mathbf{r}_3}{J}, \quad \mathbf{r}^2 = \frac{\mathbf{r}_3 \times \mathbf{r}_1}{J}, \quad \mathbf{r}^3 = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{J},$$

where, since $\mathbf{r}^\alpha \cdot \mathbf{r}_\alpha = 1$,

18.13 $$J = [\mathbf{r}_1 \ \mathbf{r}_2 \ \mathbf{r}_3].$$

Since the basic vectors $\mathbf{r}_\alpha$ are independent, $J \neq 0$. By 17.17, we have

$$[\mathbf{r}^1\,\mathbf{r}^2\,\mathbf{r}^3][\mathbf{r}_1\,\mathbf{r}_2\,\mathbf{r}_3] = \begin{vmatrix} \mathbf{r}^1\cdot\mathbf{r}_1 & \mathbf{r}^1\cdot\mathbf{r}_2 & \mathbf{r}^1\cdot\mathbf{r}_3 \\ \mathbf{r}^2\cdot\mathbf{r}_1 & \mathbf{r}^2\cdot\mathbf{r}_2 & \mathbf{r}^2\cdot\mathbf{r}_3 \\ \mathbf{r}^3\cdot\mathbf{r}_1 & \mathbf{r}^3\cdot\mathbf{r}_2 & \mathbf{r}^3\cdot\mathbf{r}_3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1,$$

so that

18.14 $$[\mathbf{r}^1\,\mathbf{r}^2\,\mathbf{r}^3] = J^{-1}$$

and the dual basic vectors are likewise independent. Interchanging "uppers and lowers" in 18.12, we obtain

18.15 $$\mathbf{r}_1 = J\mathbf{r}^2 \times \mathbf{r}^3, \quad \mathbf{r}_2 = J\mathbf{r}^3 \times \mathbf{r}^1, \quad \mathbf{r}_3 = J\mathbf{r}^1 \times \mathbf{r}^2.$$

EXERCISE

Deduce 18.14 from 18.12 without using 17.17. (*Hint:* Apply 17.13 to $\mathbf{r}_2, \mathbf{r}_3, \mathbf{r}^2, \mathbf{r}^3$.)

18.2 THE FUNDAMENTAL TENSOR

Any vector $\mathbf{u}$ may be expressed as $\sum u_\alpha \mathbf{r}^\alpha$ (meaning $u_1\mathbf{r}^1 + u_2\mathbf{r}^2 + u_3\mathbf{r}^3$) or as $\sum u^\alpha \mathbf{r}_\alpha$. The *covariant* components* u_α and *contravariant components* u^α are simply the inner products $\mathbf{u}\cdot\mathbf{r}_\alpha$ and $\mathbf{u}\cdot\mathbf{r}^\alpha$; for

$$\mathbf{u}\cdot\mathbf{r}_\beta = \sum u_\alpha \mathbf{r}^\alpha\cdot\mathbf{r}_\beta = \sum u_\alpha\,\delta^\alpha_\beta = u_\beta$$

and

$$\mathbf{u}\cdot\mathbf{r}^\beta = \sum u^\alpha \mathbf{r}_\alpha\cdot\mathbf{r}^\beta = \sum u^\alpha\,\delta^\beta_\alpha = u^\beta$$

(In such expressions as $\sum u_\alpha \mathbf{r}^\alpha\cdot\mathbf{r}_\beta$, it is understood that the summation is taken over the index α that appears twice, once "up" and once "down," and not over the index β which only appears once. The sum $\sum\delta^\alpha_\beta\, u_\alpha$ involves three values of α, one of which must be equal to β, so the "incomplete symbol" $\sum\delta^\alpha_\beta$ serves as a substitution operator transforming u_α into u_β.)

Thus

18.21 $$\mathbf{u} = \sum \mathbf{u}\cdot\mathbf{r}_\beta\,\mathbf{r}^\beta = \sum \mathbf{u}\cdot\mathbf{r}^\beta\,\mathbf{r}_\beta.$$

In particular, this holds when $\mathbf{u} = \mathbf{r}_\alpha$ or $\mathbf{r}^\alpha$, in which cases the components $u_\beta = \mathbf{u}\cdot\mathbf{r}_\beta$ and $u^\beta = \mathbf{u}\cdot\mathbf{r}^\beta$ are denoted by

18.22 $$g_{\alpha\beta} = \mathbf{r}_\alpha\cdot\mathbf{r}_\beta, \qquad g^{\alpha\beta} = \mathbf{r}^\alpha\cdot\mathbf{r}^\beta,$$

so that

18.221 $$\mathbf{r}_\alpha = \sum g_{\alpha\beta}\,\mathbf{r}^\beta, \qquad \mathbf{r}^\alpha = \sum g^{\alpha\beta}\,\mathbf{r}_\beta.$$

*For the whole story of the subtleties that underlie the terms *covariant* and *contravariant*, see Kreyszig [**1**, p. 88]. The present treatment was suggested by G. Hessenberg, Vektorielle Begründung der Differentialgeometrie, *Mathematische Annalen*, **78** (1917), 187–217.

(In the expression $\sum g_{\alpha\beta}\, \mathbf{r}^\beta$, it is β that appears twice, so we sum over β, obtaining $g_{\alpha 1}\mathbf{r}^1 + g_{\alpha 2}\mathbf{r}^2 + g_{\alpha 3}\mathbf{r}^3$.) The commutativity of inner products shows that

$$g_{\alpha\beta} = g_{\beta\alpha}, \qquad g^{\alpha\beta} = g^{\beta\alpha}.$$

The connection between the *covariant tensor* $g_{\alpha\beta}$ and the *contravariant tensor* $g^{\alpha\beta}$ is found as follows:

18.23
$$\begin{aligned}\sum g_{\alpha\gamma}\, g^{\alpha\beta} &= \sum g_{\alpha\gamma}\, \mathbf{r}^\alpha \cdot \mathbf{r}^\beta = \mathbf{r}_\gamma \cdot \mathbf{r}^\beta \\ &= \delta_\gamma^\beta.\end{aligned}$$

Thus the two symmetric matrices $\| g_{\alpha\beta} \|$ and $\| g^{\alpha\beta} \|$ have as their product the unit matrix. The two corresponding determinants cannot vanish, since their product is 1. When the coefficients $g_{\alpha\gamma}$ are given, we have, in 18.23 for each value of β, a set of three linear equations

$$g_{1\gamma}g^{1\beta} + g_{2\gamma}g^{2\beta} + g_{3\gamma}g^{3\beta} = \delta_\gamma^\beta \qquad (\gamma = 1, 2, 3)$$

to be solved for the three unknowns $g^{1\beta}, g^{2\beta}, g^{3\beta}$. By Cramer's rule [Birkhoff and MacLane **1**, p. 286], the solution is

$$g^{\alpha\beta} = (\text{cofactor of } g_{\alpha\beta} \text{ in } G)/G, \qquad G = \det(g_{\alpha\beta}).$$

In particular, if $g_{23} = g_{31} = g_{12} = 0$, we have

$$g^{\alpha\alpha} = 1/g_{\alpha\alpha}$$

and $g^{23} = g^{31} = g^{12} = 0$.

To express either kind of components of a vector $\mathbf{u}$ in terms of the other kind, we have

18.24
$$\begin{aligned}u_\alpha = \mathbf{u}\cdot\mathbf{r}_\alpha &= \mathbf{u}\cdot \sum g_{\alpha\beta}\, \mathbf{r}^\beta \\ &= \sum g_{\alpha\beta}\, u^\beta,\end{aligned}$$

and similarly, $u^\alpha = \sum g^{\alpha\beta} u_\beta$. The inner product of two vectors

$$\mathbf{u} = \sum u^\alpha \mathbf{r}_\alpha = \sum u_\alpha \mathbf{r}^\alpha \quad \text{and} \quad \mathbf{v} = \sum v^\beta \mathbf{r}_\beta = \sum v_\beta \mathbf{r}^\beta$$

may be expressed in various ways as a bilinear form:

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= \sum u^\alpha\, \mathbf{r}_\alpha\cdot\mathbf{v} = \sum u^\alpha v_\alpha, \\ \mathbf{u}\cdot\mathbf{v} &= \sum u_\alpha\, \mathbf{r}^\alpha\cdot\mathbf{v} = \sum u_\alpha v^\alpha, \\ \mathbf{u}\cdot\mathbf{v} &= \sum u^\alpha\, \mathbf{r}_\alpha \cdot \sum v^\beta\, \mathbf{r}_\beta = \sum\sum g_{\alpha\beta}\, u^\alpha v^\beta, \\ \mathbf{u}\cdot\mathbf{v} &= \sum u_\alpha\, \mathbf{r}^\alpha \cdot \sum v_\beta\, \mathbf{r}^\beta = \sum\sum g^{\alpha\beta}\, u_\alpha v_\beta.\end{aligned}$$

In particular, the length $|\mathbf{u}|$ is given by

18.25
$$\begin{aligned}|\mathbf{u}|^2 = \mathbf{u}\cdot\mathbf{u} &= \sum u_\alpha u^\alpha \\ &= \sum\sum g_{\alpha\beta}\, u^\alpha u^\beta = \sum\sum g^{\alpha\beta}\, u_\alpha u_\beta.\end{aligned}$$

Let us regard $\mathbf{u}$ and $\mathbf{v}$ as the position vectors of points having contravariant coordinates (u^1, u^2, u^3) and covariant coordinates (v_1, v_2, v_3), respectively. If $\mathbf{u}$ is fixed while $\mathbf{v}$ varies, the two equations

18.26 $$\mathbf{u}\cdot\mathbf{v} = 0, \qquad \mathbf{u}\cdot\mathbf{v} = 1$$

define respectively the plane through the origin perpendicular to $\mathbf{u}$, and the parallel plane whose distance from the origin, being the length of the projection of $\mathbf{v}$ on the unit vector $\mathbf{u}/|\mathbf{u}|$, is

$$\frac{\mathbf{u}}{|\mathbf{u}|}\cdot\mathbf{v} = \frac{1}{|\mathbf{u}|}.$$

The latter plane, passing through the inverse of (u^1, u^2, u^3) in the unit sphere

$$\mathbf{v}\cdot\mathbf{v} = 1,$$

is the *polar plane of* (u^1, u^2, u^3) with respect to the sphere. (See § 8.8.)

It is sometimes convenient to express the basic vectors in terms of Cartesian coordinates:

$$\mathbf{r}_\alpha = x_\alpha\mathbf{i} + y_\alpha\mathbf{j} + z_\alpha\mathbf{k}.$$

Then, by 18.22 and 17.151,

18.27 $$g_{\alpha\beta} = \mathbf{r}_\alpha\cdot\mathbf{r}_\beta = x_\alpha x_\beta + y_\alpha y_\beta + z_\alpha z_\beta,$$

18.28 $$J = [\mathbf{r}_1\,\mathbf{r}_2\,\mathbf{r}_3] = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix},$$

and the determinant of the fundamental tensor is

18.29 $$G = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = J^2.$$

EXERCISES

1. $\mathbf{u}\cdot\mathbf{v} = u^1 v_1 + u^2 v_2 + u^3 v_3$.
2. $|\mathbf{u}|^2 = g_{11}(u^1)^2 + g_{22}(u^2)^2 + g_{33}(u^3)^2 + 2g_{23}\,u^2u^3 + 2g_{31}\,u^3u^1 + 2g_{12}\,u^1u^2$.

Give the corresponding expression for $\mathbf{u}\cdot\mathbf{v}$.

3. Use 18.12, 18.221 and 18.15 to prove 17.12 in the form

$$(\mathbf{r}_1 \times \mathbf{r}_2) \times \mathbf{r}_3 = g_{13}\mathbf{r}_2 - g_{23}\mathbf{r}_1.$$

4. $\Sigma\, \mathbf{r}^\alpha \times \mathbf{r}_\alpha = \mathbf{0}$.
5. Express $\det(g^{\alpha\beta})$ in terms of $G = \det(g_{\alpha\beta})$.

18.3 RECIPROCAL LATTICES

It can fairly be said that the reciprocal lattice provides one of the most important tools available in the study of the diffraction of x-rays by crystals.

M. J. Buerger [**1**, p. 107]

The study of x-ray diffraction has confirmed the notion that the symmetrical appearance of a crystal is a result of the symmetrical pattern formed by its atoms or molecules. In other words, there is an infinite group of symmetry operations transforming the pattern (regarded as extending throughout the whole space) into itself. These operations may or may not include rotations, reflections, glide reflections, rotatory reflections or screw displacements, but in any case the translations contained in the symmetry group form a nonempty normal subgroup. This translation subgroup determines a lattice whose unit cell contains one or more atoms. The arrangement of atoms in the first cell determines their arrangement in all the other cells derived by the translations. If the cell contains only one atom, we naturally choose the origin at the center of such an atom; then instead of an atom inside each cell, we shall have one at every vertex, that is, at every lattice point. But if the cell contains several atoms there will be several superposed lattices of atoms. Thus a given crystal has a perfectly definite translation group, and the lattice becomes definite as soon as we have chosen an origin (at the center of an atom or elsewhere). There is still a theoretically unlimited choice of unit cells, though in practice we tend to use basic vectors of roughly equal lengths. However, the volume of the unit cell is definite, since it depends on the number of lattice points in a crystal of a given size. In fact, any three independent vectors generate a parallelepiped, and this is a unit cell whenever it has a lattice point at each of its eight vertices but none on its edges, nor on its faces, nor inside.

The affine theory shows that a sequence of "rational planes" 13.93 can be chosen in infinitely many ways. In Euclidean geometry these are no longer indistinguishable: each sequence has its *interplanar spacing,* which can be measured as the distance from the origin to the "first" plane

18.31 $$Xx + Yy + Zz = 1.$$

Each sequence of planes contains all the lattice points. Hence, when we compare one such sequence with another, the interplanar spacing is proportional to the density of distribution of lattice points in one plane of the sequence. This idea is physically important because the face planes and cleavage planes of a crystal naturally tend to occur among the rational planes of high density. Accordingly, we are chiefly interested in the sequences that have a relatively large interplanar spacing. On the other hand, the most interesting visible points are those at a relatively small distance from the origin. The two lattices which we are going to consider are related in such

a way that the visible points of either are in directions perpendicular to the rational planes of the other and at distances reciprocal to the interplanar spacings [Buerger **1**, p. 117]. Thus the more important planes of either lattice will correspond to the more important points of the other.

The definition in terms of a basis is extremely simple [Coxeter **1**, p. 181] and the result is easily seen to be independent of the chosen basis. If a given lattice consists of the points whose contravariant coordinates are integers, the *reciprocal* lattice consists of the points whose covariant coordinates are integers. In other words, the position vectors are

$$\mathbf{u} = \Sigma\, u^\alpha\, \mathbf{r}_\alpha \quad \text{and} \quad \mathbf{v} = \Sigma\, v_\alpha\, \mathbf{r}^\alpha,$$

respectively, where u^α and v_α are integers and $\mathbf{r}^\alpha \cdot \mathbf{r}_\beta = \delta^\alpha_\beta$. The equation $\mathbf{u} \cdot \mathbf{v} = 1$ or

$$u^1 v_1 + u^2 v_2 + u^3 v_3 = 1$$

(see 18.26) implies that the three u's are coprime, and likewise the three v's. For each visible point (u^1, u^2, u^3) of the given lattice, this equation may be identified with 18.31: it represents a first rational plane of the reciprocal lattice; perpendicular to $\mathbf{u}$ and at the reciprocal distance $|\mathbf{u}|^{-1}$. Since the distinction between "covariant" and "contravariant" is made by an arbitrary choice, the relation between the two lattices is symmetric: the first rational planes of either are the polar planes (with respect to the unit sphere) of the visible points of the other.

The shapes of the unit cells of the two lattices are determined by the inner products 18.22. For the edge lengths of these parallelepipeds, it is convenient to use the abbreviations

18.32 $$g_\alpha = \sqrt{g_{\alpha\alpha}} = |\mathbf{r}_\alpha|, \quad g^\alpha = \sqrt{g^{\alpha\alpha}} = |\mathbf{r}^\alpha|,$$

so that the angles between pairs of adjacent edges are the angles whose cosines are

$$\frac{g_{23}}{g_2 g_3},\ \frac{g_{31}}{g_3 g_1},\ \frac{g_{12}}{g_1 g_2};\ \frac{g^{23}}{g^2 g^3},\ \frac{g^{31}}{g^3 g^1},\ \frac{g^{12}}{g^1 g^2}.$$

By 18.13 and 18.14, their volumes are J and J^{-1} where, by 18.29,

$$J = \sqrt{G}, \qquad G = \det(g_{\alpha\beta}).$$

The simplest special case is the *cubic* lattice consisting of the points whose rectangular Cartesian coordinates are integers. In this case

$$\mathbf{r}_1 = \mathbf{r}^1 = \mathbf{i}, \quad \mathbf{r}_2 = \mathbf{r}^2 = \mathbf{j}, \quad \mathbf{r}_3 = \mathbf{r}^3 = \mathbf{k},$$

the distinction between covariant and contravariant disappears, and the lattice is its own reciprocal. Other important lattices are obtained as sublattices of the simple cubic lattice, that is, by putting suitable restrictions on these integral Cartesian coordinates. By making the three coordinates of each point have an even sum, we obtain the *face-centered* cubic lattice; and

by making them either all even or all odd, the *body-centered* cubic lattice. (These names refer to a larger simple lattice whose points have only even coordinates. In this larger lattice the center of a face has two odd coordinates and the center of a cell or "body" has three odd coordinates.) Each of these two lattices is similar to the reciprocal of the other; for, the bases

18.33 $\mathbf{r}_1 = (0, 1, 1), \quad \mathbf{r}_2 = (1, 0, 1), \quad \mathbf{r}_3 = (1, 1, 0),$

18.34 $\mathbf{r}^1 = (-1, 1, 1), \quad \mathbf{r}^2 = (1, -1, 1), \quad \mathbf{r}^3 = (1, 1, -1)$

evidently satisfy a trivially modified form of 18.11, namely,

$$\mathbf{r}^\alpha \cdot \mathbf{r}_\beta = 2\,\delta^\alpha_\beta .$$

This means that they are reciprocal with respect to a sphere of radius $\sqrt{2}$. To make them reciprocal with respect to a unit sphere, we merely have to divide all the coordinates of one lattice by 2 (or all the coordinates of both by $\sqrt{2}$).

For comparison with Buerger [**1**, pp. 117–127], it is perhaps worth while to point out that his

$$\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{a}^*, \mathbf{b}^*, \mathbf{c}^*, a, \ b, \ c, \ a^*, b^*, c^*, \ d_{(hkl)}, \ \sigma_{hkl}, \ V, \ V^*$$

are our

$$\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}^1, \ \mathbf{r}^2, \ \mathbf{r}^3, g_1, g_2, g_3, g^1, g^2, g^3, \ |\mathbf{v}|^{-1}, \ \mathbf{v}, \qquad J, J^{-1}.$$

EXERCISES

1. Consider two plane lattices derived from one another by a quarter-turn about the origin. Exhibit them as "reciprocal lattices"

$$u^1\mathbf{r}_1 + u^2\mathbf{r}_2 \quad \text{and} \quad v_1\mathbf{r}^1 + v_2\mathbf{r}^2.$$

(*Hint*: $\mathbf{r}^1$ is perpendicular to $\mathbf{r}_2$, and $\mathbf{r}^2$ to $\mathbf{r}_1$; also $\mathbf{r}^1 \cdot \mathbf{r}_1 = \mathbf{r}^2 \cdot \mathbf{r}_2$.)

2. Write out the fundamental tensors for the face-centered and body-centered cubic lattices with bases 18.33 and 18.34. Sketch the unit cells, which are rhombohedra. (The former may be regarded as a solid octahedron with a regular tetrahedron stuck on to each of two opposite faces.)

3. A lattice, in three dimensions as in two (§ 4.1), has a Dirichlet region (or Voronoi polyhedron) consisting of all the points that are as near to the origin as to any other lattice point. For the simple cubic lattice, this is a cube; for the face-centered lattice it is a rhombic dodecahedron, whose faces are twelve equal rhombi; and for the body-centered lattice it is a truncated octahedron, whose faces consist of six squares and eight regular hexagons [Steinhaus **2**, pp. 152, 157].

18.4 THE CRITICAL LATTICE OF A SPHERE

Given a bounded region K . . . which contains the origin O as an inner point, we consider all those lattices which have no point except O in the interior of K. The lower bound of their determinants is called the critical determinant of K . . . and the lattices for which this lower bound is attained are called the critical lattices of K.

Harold Davenport (1907-1969)*

Every lattice has a certain minimum distance c between pairs of lattice points, and a certain volume J for its unit cell (§ 13.9). This J is called the *determinant* of the lattice, because it is the determinant of the Cartesian components of the three basic vectors $\mathbf{r}_\alpha$. We proceed to prove that, for a given value of c, the minimum value of J occurs when the lattice is face-centered cubic.†

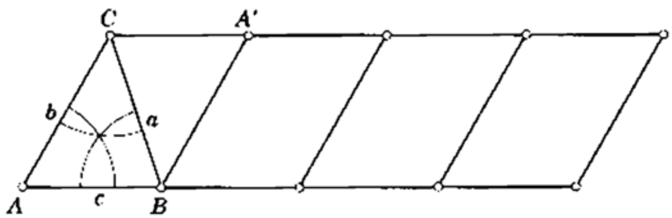


Figure 18.4a

Consider any point A of a given lattice whose unit cell has volume J. Choose a lattice point B at the minimum distance c from A, and a lattice point C outside the line AB, at the shortest distance b ($\geqslant c$) from A. These points can always be chosen so that the angle A and sides a, b, c of the triangle ABC satisfy

$$A \leqslant \tfrac{1}{2}\pi, \qquad a \geqslant b \geqslant c,$$

and therefore
$$b^2 + c^2 \geqslant a^2$$

(Figure 18.4a). Let Δ and R denote the area and circumradius of this triangle, so that, by 1.53 and 1.55,

$$16\Delta^2 = -a^4 - b^4 - c^4 + 2b^2c^2 + 2c^2a^2 + 2a^2b^2, \quad 16\Delta^2R^2 = b^2c^2a^2.$$

The plane ABC is, of course, a rational plane of the lattice. In one of the two nearest parallel planes of the same system, there is a lattice point D

* Recent progress in the geometry of numbers, *Proceedings of the International Congress of Mathematicians*, 1950, vol. I, p. 166.

† A. P. Dempster, The minimum of a definite ternary quadratic form, *Canadian Journal of Mathematics*, **9** (1957), pp. 232–234.

whose orthogonal projection D_1 on the plane ABC is not outside the parallelogram $ABA'C$. Replacing the triangle ABC, if necessary, by the triangle $A'BC$, we may assume that D_1 is not outside the triangle ABC. (The possible change of notation involves the central inversion that interchanges B and C.) Denoting by d the distance DD_1 from D to the plane ABC, we have

$$J = 2\Delta d.$$

Since none of AD, BD, CD is parallel to AB, all of them are greater than or equal to the next shortest distance b. Since the triangle ABC has no obtuse angle, circles of radius R with centers at the vertices overlap in such a way that every interior point of the triangle except the circumcenter is inside at least one of these circles. Therefore the distance of D_1 from at least one vertex is less than R, except that it is equal to R when D_1 is the circumcenter. Thus at least one of AD, BD, CD has its square less than or equal to $R^2 + d^2$, and consequently

$$R^2 + d^2 \geqslant b^2,$$

with equality only when D_1 is the circumcenter (and possibly not even then). Hence

$$\begin{aligned} J^2 &= (2\Delta d)^2 \\ &\geqslant 4\Delta^2(b^2 - R^2) = \tfrac{1}{4}b^2(-a^4 - b^4 - c^4 + 2b^2c^2 + c^2a^2 + 2a^2b^2) \\ &= \tfrac{1}{2}c^6 + \tfrac{1}{4}c^2(b^2 - c^2)(3b^2 + 2c^2) + \tfrac{1}{4}b^2(a^2 - b^2)(b^2 + c^2 - a^2) \\ &\geqslant \tfrac{1}{2}c^6, \end{aligned}$$

with equality only when

$$R^2 + d^2 = b^2, \qquad b = c,$$

and either

$$\text{(i) } a = b \quad \text{or} \quad \text{(ii) } b^2 + c^2 = a^2.$$

Thus there are apparently two "critical" lattices for which the ratio J/c^3 attains its minimum value $\sqrt{\tfrac{1}{2}}$. However, we shall soon see that these two are merely different aspects of one: the face-centered cubic lattice.

In case (i) (Figure 18.4b) the tetrahedron $ABCD$ is regular, and we may choose Cartesian coordinates

$$(0, 0, 0), \quad (0, 1, 1), \quad (1, 0, 1), \quad (1, 1, 0)$$

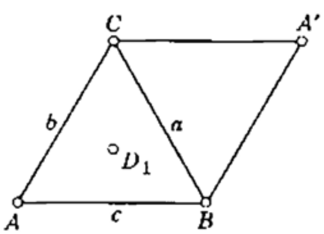


Figure 18.4b

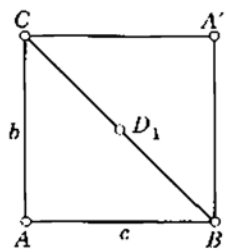


Figure 18.4c

for its vertices, in agreement with the basis 18.33 for the face-centered cubic lattice. In case (ii) (Figure 18.4c) $ABA'C$ is a square, the base of a pyramid whose sloping faces (such as ABD) are equilateral triangles. Choosing A, B and D as before, we now have C at $(0, 1, -1)$, yielding the alternative basis

18.41 $$\mathbf{r}_1 = (0, 1, 1), \quad \mathbf{r}_3 - \mathbf{r}_2 = (0, 1, -1), \quad \mathbf{r}_3 = (1, 1, 0)$$

for the same lattice. Thus we have proved that the face-centered cubic lattice (whose points have integral Cartesian coordinates with an even sum) is really the only "critical" lattice.

By 18.25, the square of the length of the lattice vector

$$\mathbf{u} = \Sigma\, u^\alpha \mathbf{r}_\alpha$$

is

$$\Sigma\Sigma\, g_{\alpha\beta}\, u^\alpha u^\beta,$$

and c^2 is the minimum value of this positive definite ternary quadratic form when the coordinates u^1, u^2, u^3 are restricted to integral values other than 0, 0, 0. Hence, among all such forms with a given minimum value c^2, the minimum determinant $G = J^2 = \frac{1}{2}c^6$ occurs when the basic vectors are given by 18.33, so that the form is

$$\begin{aligned}(\Sigma\, u^\alpha \mathbf{r}_\alpha)^2 &= (u^2 + u^3,\ u^3 + u^1,\ u^1 + u^2)^2 \\ &= (u^2 + u^3)^2 + (u^3 + u^1)^2 + (u^1 + u^2)^2 \\ &= 2\{(u^1)^2 + (u^2)^2 + (u^3)^2 + u^2u^3 + u^3u^1 + u^1u^2\}.\end{aligned}$$

In other words, every "extreme" form in three variables is equivalent to

$$(u^1)^2 + (u^2)^2 + (u^3)^2 + u^2u^3 + u^3u^1 + u^1u^2.$$

This is a famous result, first proved by Gauss [**1**, vol. 2, pp. 192–196].*

EXERCISE

Using the basis 18.41 instead of 18.33, obtain the equivalent form

$$(u^1)^2 + (u^2)^2 + (u^3)^2 + u^2u^3 + u^3u^1.$$

18.5 GENERAL COORDINATES

The position of a point in Euclidean space may be specified by three numbers in many ways. Rectangular Cartesian coordinates (x, y, z) are the most familiar; but we have seen (e.g., in § 17.7) that other systems, such as cylindrical coordinates, are sometimes more convenient. Let us use the notation (u^1, u^2, u^3) for general coordinates. The essential requirements are that, within a suitable range of variation, x, y, z are single-valued differentiable functions of u^1, u^2, u^3, while u^1, u^2, u^3 are equally well-behaved functions of x, y, z. For instance, if (u^1, u^2, u^3) are cylindrical coordinates, we have

* For an account of the history of extreme forms up to 1951, see Coxeter, *Canadian Journal of Mathematics*, **3** (1951), p. 393.

$$x = u^1 \cos u^2, \qquad y = u^1 \sin u^2, \qquad z = u^3;$$
$$u^1 = \sqrt{x^2 + y^2}, \qquad u^2 = \arctan \frac{y}{x}, \qquad u^3 = z.$$

For arbitrary constants a, b, c, the surfaces

18.51 $$u^1 = a, \qquad u^2 = b, \qquad u^3 = c$$

are called *level surfaces,* and their three curves of intersection such as

$$u^2 = b, \quad u^3 = c$$

are called *level curves.* Through a given point there is usually one level surface of each kind, and one level curve of each kind, but exceptions are allowed. For instance, in the case of cylindrical coordinates the level surfaces are cylinders $u^1 = a$ or $x^2 + y^2 = a^2$, planes $u^2 = b$ or $y = x \tan b$ through the z-axis, and planes $z = c$ orthogonal to the z-axis. The z-axis itself is exceptional because each of its points lies on infinitely many planes $u^2 = b$ (in fact, on all of them).

According to the ordinary meaning of partial differentiation, the partial derivatives of the position vector

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

are the unit vectors along the Cartesian axes:

$$\frac{\partial \mathbf{r}}{\partial x} = \mathbf{i}, \quad \frac{\partial \mathbf{r}}{\partial y} = \mathbf{j}, \quad \frac{\partial \mathbf{r}}{\partial z} = \mathbf{k}.$$

The differential of $\mathbf{r}$, representing displacement in any given direction, is

$$d\mathbf{r} = \mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz = (dx, dy, dz),$$

and the element of arc of any curve in this direction is ds, where

18.52 $$(ds)^2 = |d\mathbf{r}|^2 = d\mathbf{r}\cdot d\mathbf{r} = (dx)^2 + (dy)^2 + (dz)^2.$$

Instead of regarding $\mathbf{r}$ as a function of x, y, z, we may regard it as a function of u^1, u^2, u^3. Using a subscript α to indicate a partial derivative with respect to u^α (so that $x_\alpha = \partial x/\partial u^\alpha$, etc.), we have

18.53 $$\mathbf{r}_\alpha = \frac{\partial \mathbf{r}}{\partial u^\alpha} = x_\alpha \mathbf{i} + y_\alpha \mathbf{j} + z_\alpha \mathbf{k}$$

and $$d\mathbf{r} = \mathbf{r}_1\,du^1 + \mathbf{r}_2\,du^2 + \mathbf{r}_3\,du^3 = \Sigma\, \mathbf{r}_\alpha\, du^\alpha.$$

For a displacement along the level curve $u^2 = b$, $u^3 = c$ we have $du^2 = 0$, $du^3 = 0$, and $d\mathbf{r} = \mathbf{r}_1\,du^1$. Thus $\mathbf{r}_1$ is a *tangent* vector to this level curve. Similarly $\mathbf{r}_2$ is a tangent vector to the curve $u^3 = c$, $u^1 = a$, and $\mathbf{r}_3$ is a tangent vector to $u^1 = a$, $u^2 = b$. At a general point in space we thus have a definite trihedron $\mathbf{r}_1\,\mathbf{r}_2\,\mathbf{r}_3$ depending on the coordinate system. These basic vectors are not necessarily of unit length, and not necessarily orthogonal (though they do happen to be orthogonal in the case of cylindrical coordi-

nates). The derivatives of the Cartesian coordinates are expressible in terms of them:

18.54 $$x_\alpha = \mathbf{r}_\alpha \cdot \mathbf{i}, \quad y_\alpha = \mathbf{r}_\alpha \cdot \mathbf{j}, \quad z_\alpha = \mathbf{r}_\alpha \cdot \mathbf{k}.$$

At a general point in space, any two of the three basic vectors determine a tangent plane to one of the three level surfaces through the point. For instance, the plane $\mathbf{r}_2\mathbf{r}_3$ touches the surface $u^1 = a$ since it contains tangents to two curves lying in that surface. Hence the dual basic vectors 18.12, orthogonal to the tangent planes $\mathbf{r}_2\mathbf{r}_3$, $\mathbf{r}_3\mathbf{r}_1$, $\mathbf{r}_1\mathbf{r}_2$, are *normals* to the level surfaces 18.51: not, in general, of unit length, but adjusted so that

$$\mathbf{r}^\alpha \cdot \mathbf{r}_\alpha = 1 \qquad (\alpha = 1, 2, 3).$$

The notation 18.22 provides the following formula for the element of arc ds in the direction of a given displacement $d\mathbf{r}$:

18.55 $$\begin{aligned}(ds)^2 &= d\mathbf{r} \cdot d\mathbf{r} = \Sigma\, \mathbf{r}_\alpha\, du^\alpha \cdot \Sigma\, \mathbf{r}_\beta\, du^\beta \\ &= \Sigma\Sigma\, g_{\alpha\beta}\, du^\alpha\, du^\beta \\ &= g_{11}\,(du^1)^2 + g_{22}\,(du^2)^2 + g_{33}\,(du^3)^2 \\ &\qquad + 2g_{23}\,du^2\,du^3 + 2g_{31}\,du^3\,du^1 + 2g_{12}\,du^1\,du^2.\end{aligned}$$

In the special case when u^1, u^2, u^3 are x, y, z, this reduces to 18.52. In general, the coefficients $g_{\alpha\beta}$ are not constants but functions of the coordinates and their derivatives (see 18.27).

To deal with any given system of coordinates, we work out $g_{\alpha\beta} = \mathbf{r}_\alpha \cdot \mathbf{r}_\beta$ from the derivatives 18.53, then obtain $g^{\alpha\beta}$ by taking the cofactor of $g_{\alpha\beta}$ in the determinant G and dividing by G itself.

Our use of the letter J in 18.13 and 18.28 commemorates the German mathematician C. G. J. Jacobi (1804–1851). In fact, for transforming the triple integral of a function

$$f(x, y, z) = F(u^1, u^2, u^3)$$

from Cartesian to other coordinates, we use the formula

$$\iiint f(x, y, z)\, dx\, dy\, dz = \iiint F(u^1, u^2, u^3)\, \frac{\partial(x, y, z)}{\partial(u^1, u^2, u^3)}\, du^1\, du^2\, du^3,$$

which involves the *Jacobian*

$$\frac{\partial(x, y, z)}{\partial(u^1, u^2, u^3)} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = [\mathbf{r}_1\, \mathbf{r}_2\, \mathbf{r}_3] = J.$$

EXERCISES

1. If u^1, u^2, u^3 are *affine* coordinates, they are the components of $\mathbf{r}$ with reference to three fixed independent vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, so that

18.56 $$\mathbf{r} = \Sigma\, u^\alpha\, \mathbf{r}_\alpha.$$

(This notation is appropriate since it makes $\mathbf{r}_\alpha = \partial\mathbf{r}/\partial u^\alpha$.) The components of $\mathbf{r}$ with reference to $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are

$$x = \Sigma\, x_\alpha u^\alpha, \qquad y = \Sigma\, y_\alpha u^\alpha, \qquad z = \Sigma\, z_\alpha u^\alpha;$$

$\mathbf{r}_\alpha = x_\alpha\mathbf{i} + y_\alpha\mathbf{j} + z_\alpha\mathbf{k}$ is a constant vector for each α, and all the $g_{\alpha\beta}$ are constants.

2. *Oblique* Cartesian coordinates are affine coordinates with the same unit of measurement along all three axes, so that $|\mathbf{r}_\alpha| = 1$. In this case $g_{\alpha\alpha} = 1$, and $g_{\alpha\beta}$ is the cosine of the angle between $\mathbf{r}_\alpha$ and $\mathbf{r}_\beta$ (which are the axes of the coordinates u^α and u^β).

3. *Rectangular* Cartesian coordinates (with axes rotated to new positions without changing the origin) arise when $\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3$ is an orthogonal trihedron of unit vectors, like $\mathbf{i}\,\mathbf{j}\,\mathbf{k}$, so that

18.57 $$g_{\alpha\beta} = \delta_{\alpha\beta}$$

(meaning 1 or 0 according as α and β are equal or unequal). By 18.54, x_α is the cosine of the angle between the new axis $\mathbf{r}_\alpha$ and the old axis $\mathbf{i}$; similarly for y_α and z_α. Interchanging the roles of the new and old axes in the relation

$$\mathbf{r}_\alpha = x_\alpha\mathbf{i} + y_\alpha\mathbf{j} + z_\alpha\mathbf{k},$$

we obtain

18.58 $$\mathbf{i} = \Sigma\, x_\alpha\, \mathbf{r}_\alpha, \qquad \mathbf{j} = \Sigma\, y_\alpha\, \mathbf{r}_\alpha, \qquad \mathbf{k} = \Sigma\, z_\alpha\, \mathbf{r}_\alpha.$$

(Since now $\mathbf{r}^\alpha = \mathbf{r}_\alpha$, the distinction between covariant and contravariant disappears.) From 18.56 deduce

$$u^\alpha = \mathbf{r}_\alpha \cdot \mathbf{r} = x_\alpha x + y_\alpha y + z_\alpha z$$

so that (paradoxically)

$$\frac{\partial u^\alpha}{\partial x} = x_\alpha = \frac{\partial x}{\partial u^\alpha}.$$

From 18.27 and 18.57 deduce

$$x_\alpha x_\beta + y_\alpha y_\beta + z_\alpha z_\beta = \delta_{\alpha\beta}.$$

With the help of 18.58, evaluate $\Sigma\, x_\alpha{}^2$ and $\Sigma\, y_\alpha z_\alpha$. (In technical language, these properties make

$$\begin{Vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{Vmatrix}$$

an "orthogonal matrix.")

4. Find $g_{\alpha\beta}$ in the case of *cylindrical* coordinates. Verify that $G = J^2$.

5. Find $g_{\alpha\beta}$ in the case of *spherical polar* coordinates, defined by

$$x = u^3 \sin u^1 \cos u^2, \quad y = u^3 \sin u^1 \sin u^2, \quad z = u^3 \cos u^1.$$

Describe the level surfaces.

6. Find $g_{\alpha\beta}$ in the case of *confocal* coordinates, defined by

$$x^2 = \frac{(A - u^1)(A - u^2)(A - u^3)}{(A - B)(A - C)}, \quad y^2 = \frac{(B - u^1)(B - u^2)(B - u^3)}{(B - C)(B - A)},$$
$$z^2 = \frac{(C - u^1)(C - u^2)(C - u^3)}{(C - A)(C - B)},$$

where $u^1 < C < u^2 < B < u^3 < A$ (and x^2 means "x squared"). In this case the level surfaces are the central quadrics

18.59
$$\frac{x^2}{A-\lambda}+\frac{y^2}{B-\lambda}+\frac{z^2}{C-\lambda}=1,$$

which are ellipsoids $u^1 = \lambda$ if $\lambda < C$, one-sheet hyperboloids $u^2 = \lambda$ if $C < \lambda < B$, and two-sheet hyperboloids $u^3 = \lambda$ if $B < \lambda < A$. In fact, u^1, u^2, u^3 are the roots of 18.59, regarded as a cubic equation for λ [Weatherburn **2**, p. 211].

18.6 THE ALTERNATING SYMBOL

As a kind of counterpart of the Kronecker delta, we shall find it convenient to use the "alternating epsilon"

$$\varepsilon^{\alpha\beta\gamma} = \varepsilon_{\alpha\beta\gamma} = \tfrac{1}{2}(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta),$$

which is 1 if $\alpha\beta\gamma$ is an even permutation of 123, -1 if it is an odd permutation, and 0 otherwise. This trick provides one of the best ways to introduce the theory of determinants [Jeffreys **1**, p. 13]:

18.61
$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \Sigma\Sigma\Sigma\, \varepsilon^{\alpha\beta\gamma}\, x_\alpha y_\beta z_\gamma,$$

$$\begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} = \Sigma\Sigma\Sigma\, \varepsilon^{\alpha\beta\gamma}\, g_{1\alpha}\, g_{2\beta}\, g_{3\gamma}.$$

From 18.12–18.15 we deduce

$$[\mathbf{r}_\alpha\, \mathbf{r}_\beta\, \mathbf{r}_\gamma] = \varepsilon_{\alpha\beta\gamma} J, \qquad [\mathbf{r}^\alpha\, \mathbf{r}^\beta\, \mathbf{r}^\gamma] = \varepsilon^{\alpha\beta\gamma} J^{-1},$$
$$J^{-1}\, \mathbf{r}_\alpha \times \mathbf{r}_\beta = \Sigma\, \varepsilon_{\alpha\beta\gamma}\, \mathbf{r}^\gamma, \qquad J\, \mathbf{r}^\alpha \times \mathbf{r}^\beta = \Sigma\, \varepsilon^{\alpha\beta\gamma}\, \mathbf{r}_\gamma.$$

Since $\Sigma\Sigma\, \varepsilon^{\alpha\beta\gamma}\, g_{\alpha\beta} = 0$, it follows that

$$\begin{aligned} \Sigma\, \mathbf{r}^\alpha \times \mathbf{r}_\alpha &= \Sigma\, \mathbf{r}^\alpha \times \Sigma\, g_{\alpha\beta} \mathbf{r}^\beta = \Sigma\Sigma\Sigma\, \varepsilon^{\alpha\beta\gamma}\, g_{\alpha\beta}\, \mathbf{r}_\gamma / J \\ &= \mathbf{0}. \end{aligned}$$

The analogous "two-dimensional" symbol is

$$\varepsilon^{ij} = \varepsilon_{ij} = j - i \qquad (i = 1 \text{ or } 2,\; j = 1 \text{ or } 2)$$

which enables us to write

$$\begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} = \Sigma\Sigma\, \varepsilon^{ij}\, g_{1i}\, g_{2j}.$$

(We use Latin or Greek indices according as the range of values is 12 or 123.)

EXERCISES

1. Use 18.61 to obtain a formula for the cofactor of x_α. Work this out for the case when $\alpha = 3$.

2. If $c^{ij} = c^{ji}$, $\Sigma\Sigma\, \varepsilon_{ij}\, c^{ij} = 0$. Use the same idea to justify the step $\Sigma\Sigma\, \varepsilon^{\alpha\beta\gamma}\, g_{\alpha\beta} = 0$ i h bove evaluation of $\Sigma\, \mathbf{r}^\alpha \times \mathbf{r}_\alpha$.

19

Differential geometry of surfaces

The present chapter extends the notion of *curvature* from curves to surfaces. This extension is achieved by considering plane sections of a given surface, especially normal sections. Through the normal at a given point we can draw infinitely many planes; in fact, we can imagine such a plane to rotate continuously about the normal. In general, the curvature of the section varies continuously. For one of the planes the curvature attains its maximum value, for another, its minimum. We shall see that these two planes are at right angles, and that the product of the two "principal curvatures" determines the essential nature of the surface. For instance, this "Gaussian curvature" is positive for an oval surface such as an ellipsoid, zero for a developable surface such as a cylinder or cone, and negative for a saddle-shaped surface such as a hyperbolic paraboloid.

19.1 THE USE OF TWO PARAMETERS ON A SURFACE

> *To fix the position of a point on the earth's surface, we may give its latitude and longitude. . . . Through points on the equator draw meridians; through points on the Greenwich meridian, draw parallels of latitude. The position of a point . . . is then given by the two curves, one of each family, which go through it. . . . Each point, except the poles, acquires two definite coordinates. We can generalize this method to any surface, or rather to a piece of any surface; we take two families of curves on the surface, such that through each point goes just one curve of each family . . . as if a fine fishing-net were thrown over the surface.*
>
> H. G. Forder [**3,** p. 133]

A surface $f(x, y, z) = 0$ is often conveniently represented by a set of three parametric equations

$$x = x(u^1, u^2), \quad y = y(u^1, u^2), \quad z = z(u^1, u^2),$$

from which the single equation $f = 0$ could be derived by eliminating the

parameters u^1, u^2. We shall assume, as before, that the functions involved are continuous and possess all the continuous derivatives that we need.

A simple instance arises when we regard x and y as the parameters, so that the three equations become

$$x = u^1, \quad y = u^2, \quad z = F(u^1, u^2),$$

where the expression for z is the result of solving the equation $f(x, y, z) = 0$ for z in terms of x and y. Such an equation

$$z = F(x, y)$$

is called *Monge's form* of the equation for a surface. For instance, the sphere $x^2 + y^2 + z^2 = 1$ becomes

$$z = \pm\sqrt{1 - x^2 - y^2}.$$

The square root makes this a clumsy way to investigate the sphere. It is far better to take u^1 and u^2 to be colatitude and longitude, so that

19.11 $$x = \sin u^1 \cos u^2, \qquad y = \sin u^1 \sin u^2, \qquad z = \cos u^1.$$

(Colatitude u^1 means latitude $\frac{1}{2}\pi - u^1$.)

The vector equation for a surface is

19.12 $$\mathbf{r} = \mathbf{r}(u^1, u^2),$$

just as the vector equation for a curve is $\mathbf{r} = \mathbf{r}(u)$. The essential difference is that the curve has only one parameter, whereas the surface has two independent parameters.

One way to explore a surface is to investigate families of curves lying on it. Among these are the *parametric* curves

$$u^1 = a \quad \text{and} \quad u^2 = b,$$

where a and b are arbitrary constants. Through a given point there is usually one parametric curve of each kind, but exceptions are allowed. For instance, when colatitude and longitude are used on the unit sphere, the curves $u^1 = a$ are circles called *parallels of latitude* and the curves $u^2 = b$ are great circles called *meridians*. Almost every point on the sphere lies on one "parallel" and one meridian, but the north and south poles lie on all the meridians.

The position vector $\mathbf{r}$, of a point on the surface, is a vector function of u^1 and u^2. Using a subscript i to indicate a partial derivative with respect to u^i, we have

$$d\mathbf{r} = \mathbf{r}_1\, du^1 + \mathbf{r}_2\, du^2 = \sum \mathbf{r}_i\, du^i,$$

where $$\mathbf{r}_i = \frac{\partial \mathbf{r}}{\partial u^i}.$$

The differential $d\mathbf{r}$ may be regarded as a displacement along a given curve on the surface or, more precisely, a displacement along a tangent. In the

case of the parametric curve $u^2 = b$, we have $du^2 = 0$, so that $d\mathbf{r} = \mathbf{r}_1\, du^1$. Thus $\mathbf{r}_1$ is a tangent vector to this curve, and similarly $\mathbf{r}_2$ is a tangent vector to the other parametric curve $u^1 = a$. It follows that the plane $\mathbf{r}_1\mathbf{r}_2$, spanned by these two covariant basic vectors, is the *tangent plane* to the surface at the point considered, which is the point $(u^1, u^2) = (a, b)$. In the same plane we define the two contravariant basic vectors $\mathbf{r}^i$ to be normal to the parametric curves, adjusted so that

$$\mathbf{r}^i \cdot \mathbf{r}_j = \delta^i_j.$$

In the tangent plane, there are tangent vectors going out from the point of contact in all directions. Any such vector

19.13 $$\mathbf{t} = \Sigma\, a_i\, \mathbf{r}^i = \Sigma\, a^i\, \mathbf{r}_i$$

is said to have covariant components a_i and contravariant components a^i. It is easily verified (cf. § 18.2) that

19.14 $$a_j = \mathbf{t} \cdot \mathbf{r}_j, \qquad a^j = \mathbf{t} \cdot \mathbf{r}^j.$$

In particular, the basic vectors themselves have components g_{ij}, g^{ij}, such that

19.15 $$\mathbf{r}_i = \Sigma\, g_{ij}\, \mathbf{r}^j, \qquad \mathbf{r}^i = \Sigma\, g^{ij}\, \mathbf{r}_j,$$
$$g_{ij} = \mathbf{r}_i \cdot \mathbf{r}_j, \qquad g^{ij} = \mathbf{r}^i \cdot \mathbf{r}^j.$$

In terms of the *fundamental magnitudes of the first order*

$$g_{11}, \quad g_{12} = g_{21}, \quad g_{22},$$

defined by $g_{ij} = \mathbf{r}_i \cdot \mathbf{r}_j$, we have the following formula for the element of arc ds (of any curve on the surface):

19.16 $$\begin{aligned} ds^2 &= d\mathbf{r} \cdot d\mathbf{r} = \Sigma\, \mathbf{r}_i\, du^i \cdot \Sigma\, \mathbf{r}_j\, du^j \\ &= \Sigma\Sigma g_{ij}\, du^i\, du^j \\ &= g_{11}(du^1)^2 + 2g_{12}\, du^1\, du^2 + g_{22}(du^2)^2. \end{aligned}$$

The fundamental magnitudes (which are functions of the parameters u^i) are spoken of collectively as a covariant tensor. The corresponding contravariant tensor g^{ij} is given by the last part of 19.15, which implies

$$\Sigma\, g_{ik}\, g^{ij} = \delta^j_k.$$

For each value of j, this is a pair of equations to be solved for the two unknowns g^{ij} $(i = 1, 2)$. The solution is

$$g^{ij} = (\text{cofactor of } g_{ij} \text{ in } g)/g,$$

where $$g = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} = g_{11}g_{22} - (g_{12})^2.$$

Since the number of rows (or columns) in this determinant is only 2, the cofactors are single elements, and we have the explicit expressions

19.17 $$g^{11} = \frac{g_{22}}{g}, \quad g^{12} = g^{21} = -\frac{g_{12}}{g}, \quad g^{22} = \frac{g_{11}}{g}.$$

It is now easy to derive the covariant components of the tangent vector $\mathbf{t}$ from its contravariant components, or vice versa:

$$a_j = \mathbf{t}\cdot\mathbf{r}_j = \Sigma\, a^i\, \mathbf{r}_i\cdot\mathbf{r}_j = \Sigma\, g_{ij}\, a^i,$$
$$a^j = \mathbf{t}\cdot\mathbf{r}^j = \Sigma\, a_i\, \mathbf{r}^i\cdot\mathbf{r}^j = \Sigma\, g^{ij}\, a_i.$$

Of course, we are free to interchange i and j, obtaining

19.18 $$a_i = \Sigma\, g_{ij}\, a^j, \qquad a^i = \Sigma\, g^{ij}\, a_j.$$

Since $g_{12} = \mathbf{r}_1\cdot\mathbf{r}_2$, where $\mathbf{r}_1$ and $\mathbf{r}_2$ are tangents to the parametric curves, the condition for the two families of parametric curves to intersect at right angles is

$$g_{12} = 0.$$

Thus in the case of orthogonal parametric curves we have simply

$$g = g_{11}\, g_{22}, \quad g^{12} = 0, \quad g^{ii} = 1/g_{ii},$$

whence by 19.15,

$$\mathbf{r}^i = g^{ii}\, \mathbf{r}_i = \mathbf{r}_i/g_{ii}.$$

EXERCISES

1. The basic vector $\mathbf{r}_i$ has covariant components g_{ij}, contravariant components δ_i^j.
2. $\Sigma\, \mathbf{r}^j \times \mathbf{r}_j = \mathbf{0}$. Interpret this geometrically in terms of areas of triangles.
3. Find $\mathbf{r}^1$ and $\mathbf{r}^2$ for the general surface of revolution

$$\mathbf{r} = (u^1 \cos u^2, \quad u^1 \sin u^2, \quad z),$$

where z is a function of u^1 alone.

4. Find g_{ij} and g^{ij} for the unit sphere expressed in terms of colatitude and longitude.

19.2 DIRECTIONS ON A SURFACE

Just as a curve in the (x, y)-plane is given by an equation connecting x and y, a curve on the surface 19.12 is given by an equation connecting u^1 and u^2. A differential equation determines a family of curves. In general, a first-order, first-degree differential equation

$$\Sigma\, c_i\, du^i = 0$$

determines a one-parameter family of curves: one curve through each point of general position on the surface, going out from that point in a direction determined by $du^2/du^1 = -c_1/c_2$; for instance, the equation

$$du^2 = 0$$

determines the "first" family of parametric curves. On the other hand, a first-order, second-degree equation

19.21 $$\Sigma\Sigma\, c_{ij}\, du^i\, du^j = 0,$$

where $c_{12} = c_{21}$ and $c_{11}\, c_{22} < c_{12}{}^2$, determines a *net* of curves: two curves through a general point on the surface; for example, the quadratic equation

$$du^1\, du^2 = 0$$

determines a net consisting of the two families of parametric curves taken together.

We have seen that the vectors $\mathbf{r}_i$ are in the directions of the tangents to the parametric curves. Since

$$\mathbf{r}_i{}^2 = \mathbf{r}_i \cdot \mathbf{r}_i = g_{ii},$$

their lengths are $\sqrt{g_{ii}}$. Because of its frequent occurrence, we shall use the abbreviation g_i for this square root; thus

$$g_i = \sqrt{g_{ii}} = |\mathbf{r}_i|,$$

and analogously

$$g^i = \sqrt{g^{ii}} = |\mathbf{r}^i|.$$

In this notation, the *unit* tangent vectors to the parametric curves (touching $du^2 = 0$ and $du^1 = 0$, respectively) are

$$\mathbf{r}_1/g_1 \quad \text{and} \quad \mathbf{r}_2/g_2.$$

The angle ϕ at which the two parametric curves intersect is given by

19.22 $$\cos\phi = \frac{\mathbf{r}_1}{g_1} \cdot \frac{\mathbf{r}_2}{g_2} = \frac{g_{12}}{g_1\, g_2}.$$

We see from 19.17 that $g^1 = g_2/\sqrt{g}$, $g^2 = g_1/\sqrt{g}$; therefore

$$\sin\phi = \frac{\sqrt{g}}{g_1\, g_2} = \frac{1}{g_1\, g^1} = \frac{1}{g_2\, g^2},$$

that is, $$g_1 g^1 = g_2 g^2 = \csc\phi.$$

It follows from the definition of an outer product that the length of the vector $\mathbf{r}_1 \times \mathbf{r}_2$ is

19.23 $$|\mathbf{r}_1 \times \mathbf{r}_2| = g_1\, g_2 \sin\phi = \sqrt{g},$$

and that the element of *area* on the surface (naturally defined as the element of area in the tangent plane) is

19.24 $$dS = |\mathbf{r}_1\, du^1 \times \mathbf{r}_2\, du^2| = \sqrt{g}\, du^1\, du^2$$

[Kreyszig **1**, pp. 111–117]. The equation 19.23, in the form

$$g = (\mathbf{r}_1 \times \mathbf{r}_2)^2,$$

A displacement along any curve on the surface is given by

$$d\mathbf{r} = \Sigma\, \mathbf{r}_i\, du^i.$$

If s is the arc of this curve, the unit tangent vector is

19.25 $$\mathbf{t} = \frac{d\mathbf{r}}{ds} = \Sigma\, a^i\, \mathbf{r}_i = a^1\, \mathbf{r}_1 + a^2\, \mathbf{r}_2,$$

where

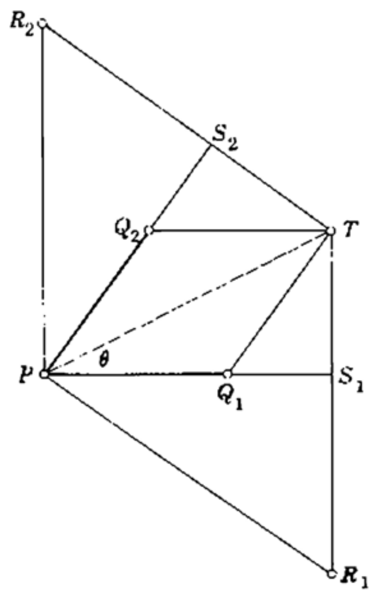
19.251 $$a^i = \frac{du^i}{ds} \qquad (i = 1, 2).$$

Thus the arc derivatives of the parameters are the contravariant components of $\mathbf{t}$. We shall not attempt to find a corresponding interpretation for the covariant components a_i, given by 19.14 or 19.18. On the other hand, it is easy to give a *geometrical* interpretation for both kinds of component. Let $\overrightarrow{PQ_1}$ and $\overrightarrow{PQ_2}$ be tangent vectors to the parametric curves, of such lengths that PT, representing $\mathbf{t}$, is a diagonal of the parallelogram PQ_1TQ_2, as in Figure 19.2*a*. Let $\mathbf{t}$ divide the angle $\phi = \angle\, Q_1PQ_2$ into the two parts θ and $\phi - \theta$. Let PR_1TR_2 be a parallelogram whose sides are perpendicular to the tangents. Since

$$\begin{aligned}\overrightarrow{PT} = \mathbf{t} &= a^1\, \mathbf{r}_1 + a^2\, \mathbf{r}_2 = \overrightarrow{PQ_1} + \overrightarrow{PQ_2} \\ &= a_1\, \mathbf{r}^1 + a_2\, \mathbf{r}^2 = \overrightarrow{PR_1} + \overrightarrow{PR_2},\end{aligned}$$

the lengths of the various lines are:

$$PQ_i = g_i\, a^i, \quad PR_i = g^i\, a_i, \quad PS_i = \mathbf{t}\cdot\mathbf{r}_i/g_i = a_i/g_i.$$



The angles are given by

$$\cos\theta = PS_1, \qquad \cos(\phi - \theta) = PS_2.$$

By taking the inner product of 19.13 with itself in various ways, we can express the obvious relation $\mathbf{t}^2 = 1$ in the equivalent forms

19.26 $$\Sigma\Sigma\, g^{ij} a_i a_j = \Sigma\, a_i a^i = \Sigma\Sigma\, g_{ij} a^i a^j = 1.$$

In virtue of 19.251, the last of these relations is a restatement of 19.16.

Similarly, by working out the inner product of two such unit tangent vectors

$$\Sigma\, a_i \mathbf{r}^i = \Sigma\, a^i \mathbf{r}_i \quad \text{and} \quad \Sigma\, b_j \mathbf{r}^j = \Sigma\, b^j \mathbf{r}_j,$$

we obtain various expressions for the cosine of the angle between them:

19.27 $$\Sigma\Sigma\, g^{ij} a_i b_j = \Sigma\, a_i b^i = \Sigma\, a_j b^j = \Sigma\Sigma\, g_{ij} a^i b^j = \Sigma\, a^i b_i.$$

Eliminating ds from the two equations 19.251, we obtain the differential equation

$$a^2\, du^1 - a^1\, du^2 = 0$$

for a family of curves whose typical tangent is given by 19.25. Another family, cutting all the members of the first family at right angles, has the differential equation

19.28 $$b^2\, du^1 - b^1\, du^2 = 0,$$

where, by 19.27,

$$\Sigma\Sigma\, g_{ij} a^i b^j = 0.$$

Writing this relation in the form $\Sigma\, a_j b^j = 0$ or

$$a_1 b^1 + a_2 b^2 = 0,$$

where $a_j = \Sigma\, g_{ij} a^i$, we find that the equation 19.28 is equivalent to

$$a_1\, du^1 + a_2\, du^2 = 0.$$

In other words,

The orthogonal trajectories of the curves $a^2\, du^1 - a^1\, du^2 = 0$ are the curves

$$\Sigma\, a_i\, du^i = 0.$$

It follows also that the two families of curves

$$\Sigma\, a_i\, du^i = 0, \qquad \Sigma\, b_j\, du^j = 0$$

are orthogonal if and only if

19.29 $$\Sigma\Sigma\, g^{ij} a_i b_j = 0.$$

The net of curves given by the quadratic differential equation 19.21 is an *orthogonal* net if and only if

19.291 $$\Sigma\Sigma\, g^{ij} c_{ij} = 0.$$

For, this condition is the same as 19.29 if the quadratic expression factorizes in the form

$$\Sigma\Sigma\, c_{ij}\, du^i\, du^j = \Sigma\, a_i\, du^i \cdot \Sigma\, b_j\, du^j,$$

so that

$$c_{ij} + c_{ji} = a_i b_j + a_j b_i.$$

EXERCISES

1. Use 17.13 to prove that $(\mathbf{r}_1 \times \mathbf{r}_2)^2 = g$ (cf. 19.23).
2. Express $\tan\phi$ in terms of the fundamental magnitudes g_{ij} and their determinant g.
3. In Figure 19.2*a*, $TS_1 = a^2/g^2$ and $TS_2 = a^1/g^1$.
4. Reconcile the formulas

$$\cos\theta = a_1/g_1, \quad \cos(\phi - \theta) = a_2/g_2,$$
$$\sin\theta = a^2/g^2, \quad \sin(\phi - \theta) = a^1/g^1$$

with 19.22.

5. The net of curves bisecting the angles between the parametric curves ($du^1\, du^2 = 0$) is given by the differential equation

$$g_{11}\,(du^1)^2 - g_{22}\,(du^2)^2 = 0.$$

(*Hint*: Find the condition for the parallelogram PQ_1TQ_2 to be a rhombus.)

6. Interpret the equation 19.21 in the case when $c_{11} = c_{22} = 0$. What does the condition 19.291 tell us in this case? What does it tell us about the curves described in the preceding exercise?
7. Use 19.24 to prove that the area of the unit sphere 19.11 is 4π.

19.3 NORMAL CURVATURE

The unit *normal* vector $\mathbf{n}$ at a point P on the surface is naturally defined as the unit vector perpendicular to the tangent plane in such a direction that the three vectors $\mathbf{r}_1\, \mathbf{r}_2\, \mathbf{n}$ form a right-handed trihedron. In virtue of 19.23, we have $\mathbf{r}_1 \times \mathbf{r}_2 = \sqrt{g}\, \mathbf{n}$, so that

19.31
$$\mathbf{n} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{\sqrt{g}},$$

$$[\mathbf{r}_1\, \mathbf{r}_2\, \mathbf{n}] = \sqrt{g},$$

and

19.32
$$\mathbf{r}_i \times \mathbf{r}_j = \varepsilon_{ij} \sqrt{g}\, \mathbf{n} \qquad (\varepsilon_{ij} = j - i).$$

Identifying this trihedron $\mathbf{r}_1\, \mathbf{r}_2\, \mathbf{n}$ with the $\mathbf{r}_1\, \mathbf{r}_2\, \mathbf{r}_3$ of § 18.1, we see from 18.12 that

$$\mathbf{r}^1 = \frac{\mathbf{r}_2 \times \mathbf{n}}{\sqrt{g}}, \quad \mathbf{r}^2 = \frac{\mathbf{n} \times \mathbf{r}_1}{\sqrt{g}}, \quad \mathbf{r}^3 = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{\sqrt{g}} = \mathbf{n} = \mathbf{r}_3.$$

The tangent plane at P contains a flat pencil of tangents

$$\mathbf{t} = \Sigma\, a^i \mathbf{r}_i$$

each of which determines a normal plane $\mathbf{tn}$. The section of the surface by such a plane is called a normal section; it is a plane curve whose curvature κ at P is called the *normal curvature* at P in the direction $\mathbf{t}$. Differentiating with respect to the arc s of the normal section, we obtain

$$\dot{\mathbf{t}} = \frac{d}{ds}\Sigma\, a^i \mathbf{r}_i = \Sigma \frac{da^i}{ds}\mathbf{r}_i + \Sigma\, a^i \frac{d}{ds}\mathbf{r}_i\,.$$

By 17.33, this is $\kappa\mathbf{n}$. Since $\mathbf{n}$ is perpendicular to $\mathbf{r}_i$, inner multiplication by $\mathbf{n}$ will eliminate the first sum on the right. (To differentiate $\mathbf{r}_i$ we use the operator

$$\frac{d}{ds} = \Sigma\, a^j \frac{\partial}{\partial u^j},$$

where $a^j = du^j/ds$.) Thus we are left with

$$\kappa = \kappa\mathbf{n}\cdot\mathbf{n} = \dot{\mathbf{t}}\cdot\mathbf{n} = \Sigma\left(a^i \frac{d}{ds}\mathbf{r}_i\right)\cdot\mathbf{n}$$

$$= \Sigma\left(a^i\, \Sigma\, a^j \frac{\partial}{\partial u^j}\mathbf{r}_i\right)\cdot\mathbf{n} = \Sigma\Sigma\, a^i a^j \frac{\partial^2\mathbf{r}}{\partial u^i\,\partial u^j}\cdot\mathbf{n} = \Sigma\Sigma\, a^i a^j \mathbf{r}_{ij}\cdot\mathbf{n}.$$

Introducing the important notation

19.34 $$b_{ij} = \mathbf{r}_{ij}\cdot\mathbf{n} \qquad (i, j = 1, 2)$$

we now have the simple formula

19.35 $$\kappa = \Sigma\Sigma\, b_{ij} a^i a^j$$

for the normal curvature in the direction $\Sigma\, a^i\mathbf{r}_i$. Since $\mathbf{r}_{ij}$ is a second derivative,

$$b_{ij} = b_{ji}.$$

The three functions b_{11}, b_{12}, b_{22} are known as *fundamental magnitudes of the second order*. Like those of the first order, they occur as coefficients in a quadratic differential form:

19.36 $$\begin{aligned}\kappa\, ds^2 &= \Sigma\Sigma\, b_{ij}\, du^i\, du^j\\ &= b_{11}(du^1)^2 + 2b_{12}\, du^1\, du^2 + b_{22}(du^2)^2.\end{aligned}$$

(It must be remembered that the normal curvature κ depends on the direction of the tangent, and therefore on $du^1 : du^2$.)

Differentiating the identity $\mathbf{r}_i\cdot\mathbf{n} = 0$, we obtain

$$\mathbf{r}_{ij}\cdot\mathbf{n} + \mathbf{r}_i\cdot\mathbf{n}_j = 0,$$

whence $$b_{ij} = -\mathbf{r}_i\cdot\mathbf{n}_j = -\mathbf{n}_i\cdot\mathbf{r}_j.$$

Along with the "covariant tensor" b_{ij}, we shall sometimes find it convenient to consider the "mixed tensor"

19.37 $$b_j^k = \Sigma\, g^{ik}\, b_{ij} = -\Sigma\, g^{ik}\, \mathbf{r}_i \cdot \mathbf{n}_j = -\mathbf{r}^k \cdot \mathbf{n}_j$$

and the "contravariant tensor"

19.371 $$\begin{aligned} b^{ik} &= \Sigma\, g^{ij}\, b_j^k = \Sigma\Sigma\, g^{ij}\, g^{kl}\, b_{jl} \\ &= \Sigma\Sigma\, g^{kl}\, g^{ji}\, b_{jl} = \Sigma\, g^{kl}\, b_l^i = b^{ki}. \end{aligned}$$

The derivative $\mathbf{n}_i$, being perpendicular to the normal $\mathbf{n}$, is a tangent vector, capable of being expressed as a linear combination of the basic vectors $\mathbf{r}^j$ or $\mathbf{r}_j$. Since its covariant and contravariant components are

$$\mathbf{n}_i \cdot \mathbf{r}_j = -b_{ij}, \qquad \mathbf{n}_i \cdot \mathbf{r}^j = -b_i^j,$$

the expressions are

19.38 $$\mathbf{n}_i = -\Sigma\, b_{ij}\, \mathbf{r}^j = -\Sigma\, b_i^j\, \mathbf{r}_j.$$

We have thus established the "Weingarten equations"

$$\begin{aligned} \mathbf{n}_1 &= -b_1^1\, \mathbf{r}_1 - b_1^2\, \mathbf{r}_2, \\ \mathbf{n}_2 &= -b_2^1\, \mathbf{r}_1 - b_2^2\, \mathbf{r}_2, \end{aligned}$$

which express the derivatives of the normal $\mathbf{n}$ in terms of the derivatives of the position vector $\mathbf{r}$.

EXERCISES

1. Evaluate $\mathbf{r}^1 \times \mathbf{r}^2$.
2. Work out b_{ij} for the unit sphere in terms of colatitude and longitude. Verify that the normal curvature is the same in all directions and the same at all points on the sphere. (*Hint*: Since $\mathbf{n} = \mathbf{r}$, $b_{ij} = -g_{ij}$.)

19.4 PRINCIPAL CURVATURES

Take a unit sphere and draw the radius parallel to the normal at a point P of [a given] surface. The radius meets the sphere in the spherical representation of P. Clearly we must distinguish between the two sides of the surface, and draw the normal on the selected side. By this representation, to a curve on the surface corresponds, in general, a curve on the sphere, and to a piece, a piece. But as normals to the surface at different points may be parallel, pieces on the sphere might overlap even when they correspond to non-overlapping pieces on the surface. But if we take pieces on the surface, not too large, this will not occur. . . . To a small piece round P on the surface will correspond a small piece on the sphere, and the ratio of the area of the latter to the area of the former, as these areas shrink to zero, tends to the total curvature at P.

H. G. Forder [**3**, pp. 139-140]

Consider a variable plane through the normal at a point P on a given

κ given by 19.35. In exceptional cases (e.g., at the north and south poles of a spheroid) it may happen that κ remains constant; such a point P is called an *umbilic*. If P is not an umbilic, a continuous rotation of the normal plane $\mathbf{nt}$ makes κ vary in such a way as to return to its original value as soon as a half-turn has been completed. With the help of 19.26 we may express 19.35 in the homogeneous form

$$\kappa \Sigma\Sigma\, g_{ij} a^i a^j = \Sigma\Sigma\, b_{ij} a^i a^j$$

or

19.41 $$\Sigma\Sigma(b_{ij} - \kappa g_{ij}) a^i a^j = 0.$$

This exhibits κ as a continuous function of the ratio

$$\frac{a^2}{a^1} = \frac{du^2}{du^1},$$

which determines the direction of the tangent

$$\mathbf{t} = \Sigma a^i \mathbf{r}_i.$$

In the course of its continuous variation, the normal curvature κ must attain at least one maximum and at least one minimum. We proceed to prove that there is just one of each, and that they occur in perpendicular directions. The maximum and minimum values of κ are called the *principal curvatures*, the positions of $\mathbf{t}$ in which they occur are called the *principal directions*, and the curves whose direction is always principal are called (perhaps not too happily) the *lines of curvature*.

As a temporary abbreviation, we write

$$c_{ij} = b_{ij} - \kappa g_{ij},$$

so that $c_{ij} = c_{ji}$. To find the principal curvatures and principal directions, we may differentiate 19.41 and then set $d\kappa = 0$ or, more conveniently, differentiate 19.41 regarding κ as a constant. Since b_{ij} and g_{ij} depend only on the fixed point P, this means that we differentiate

$$\Sigma\Sigma\, c_{ij} a^i a^j = 0$$

treating the coefficients c_{ij} as constants. Differentiating partially with respect to a^k, we find

$$\begin{aligned} \frac{\partial}{\partial a^k} \Sigma\Sigma\, c_{ij}\, a^i a^j &= \Sigma\Sigma\, c_{ij} \left(\frac{\partial a^i}{\partial a^k} a^j + a^i \frac{\partial a^j}{\partial a^k} \right) \\ &= \Sigma\Sigma\, c_{ij} (\delta^i_k a^j + a^i \delta^j_k) = \Sigma\, c_{kj} a^j + \Sigma\, c_{ik} a^i \\ &= \Sigma(c_{ki} + c_{ik}) a^i = 2\Sigma\, c_{ik} a^i. \end{aligned}$$

Restoring the proper expression for c_{ik}, we deduce

19.42 $$\Sigma\,(b_{ik} - \kappa g_{ik})\, a^i = 0 \qquad (k = 1, 2).$$

Multiplying by g^{jk} (and summing over k) to eliminate the coefficient g_{ik}, we obtain

19.43 $$\Sigma (b_i^j - \kappa\delta_i^j) a^i = 0,$$

that is,

19.44 $$\Sigma b_i^j a^i - \kappa a^j = 0 \qquad (j = 1, 2).$$

From these two equations we can find the principal curvatures by eliminating a^2/a^1, and the principal directions by eliminating κ.

When written out in detail, the equations are:

$$(b_1^1 - \kappa) a^1 + b_2^1 a^2 = 0,$$

$$b_1^2 a^1 + (b_2^2 - \kappa) a^2 = 0.$$

Eliminating a^2/a^1, we obtain

$$\begin{vmatrix} b_1^1 - \kappa & b_2^1 \\ b_1^2 & b_2^2 - \kappa \end{vmatrix} = 0,$$

that is,

19.45 $$\kappa^2 - \Sigma b_i^i \kappa + \det(b_i^j) = 0.$$

This quadratic equation has for its two roots the principal curvatures $\kappa_{(1)}$, $\kappa_{(2)}$, whose product and arithmetic mean are known as the *Gaussian curvature K* and the *mean curvature H*. Thus $\kappa_{(1)}$ and $\kappa_{(2)}$ are the roots of the equation

$$\kappa^2 - 2H\kappa + K = 0,$$

where

19.46 $$2H = \kappa_{(1)} + \kappa_{(2)} = \Sigma b_i^i = b_1^1 + b_2^2$$

and

19.47 $$K = \kappa_{(1)}\kappa_{(2)} = \det(b_i^j) = b_1^1 b_2^2 - b_2^1 b_1^2.$$

Since

$$gK = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} \begin{vmatrix} b_1^1 & b_2^1 \\ b_1^2 & b_2^2 \end{vmatrix} = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = b,$$

say, another expression for K is the ratio of the two fundamental determinants:

19.471 $$K = \frac{b}{g}.$$

When K is positive, the normal curvature (never going outside the range from $\kappa_{(1)}$ to $\kappa_{(2)}$) has the same sign in all directions; the tangent plane at P meets the surface "instantaneously" at P and not anywhere else in the

neighborhood of P. The surface is then said to be *synclastic* (or "oval"). Ellipsoids, elliptic paraboloids, and hyperboloids of two sheets are everywhere synclastic.

When K is negative, the normal curvature changes sign twice (during the rotation of the normal plane through a half-turn about the normal at P); therefore it is zero in the directions of two special tangents, called the *inflectional* tangents at P. The surface crosses its tangent plane, and its section by this plane is a pair of curves that cross each other at P, the two tangents at this "node" being the inflectional tangents.

A practical instance is the general shape of the ground at the top of a mountain pass. The tangent plane is the horizontal plane, which touches the curve of the footpath and cuts into the ground on both sides. The fact that the tangent section has a node is seen in a map on which contour lines are marked. The mountain pass occurs where one of the contour lines crosses itself [Hardy **1**, p. 65].

Such a surface is said to be *anticlastic* (or "saddle-shaped"). Nondegenerate ruled quadrics (namely, hyperbolic paraboloids and hyperboloids of one sheet) are everywhere anticlastic.

Surfaces more complicated than quadrics may be synclastic in some regions and anticlastic in others. Regions of the two kinds are then separated by a locus of *parabolic* points, at which $K = 0$. Hilbert and Cohn-Vossen [**1**, p. 197, Fig. 204] show a bust of Apollo on which the curves of parabolic points have been drawn. They are quite complicated, especially round the nose and mouth.

Surfaces on which $K = 0$ everywhere are said to be *developable*. Such surfaces include cones and cylinders, and also the surface traced out by the tangents of any twisted curve.

The Weingarten equations 19.38 provide a useful expression for the Gaussian curvature as a triple product:

19.48 $$K = [\mathbf{n}\ \mathbf{n}_1\ \mathbf{n}_2]/\sqrt{g}.$$

In fact,

$$\begin{aligned}[\mathbf{n}\ \mathbf{n}_1\ \mathbf{n}_2] &= [\mathbf{n}\ \ \Sigma\, b_1^j \mathbf{r}_j\ \ \Sigma\, b_2^k \mathbf{r}_k] \quad = \Sigma\Sigma\, b_1^j b_2^k [\mathbf{n}\ \mathbf{r}_j\ \mathbf{r}_k] \\ &= \Sigma\Sigma\, b_1^j b_2^k \varepsilon_{jk} \sqrt{g} \quad = \det(b_i^j)\sqrt{g} \\ &= K\sqrt{g}.\end{aligned}$$

Another expression, involving an arbitrary unit tangent vector $\mathbf{t}$, was discovered by A. J. Coleman:

19.49 $$\sqrt{g}\, K = \Sigma\Sigma\, \varepsilon^{ij}\, [\mathbf{n}\ \mathbf{t}\ \mathbf{t}_i]_j,$$

where the final subscript indicates differentiation with respect to u^j. This

is deduced from Lagrange's identity 17.13 by introducing another unit tangent vector $\mathbf{m} = \mathbf{n} \times \mathbf{t}$, so that $\mathbf{n} = \mathbf{t} \times \mathbf{m}$ and

$$\begin{aligned}[\mathbf{n}\,\mathbf{n}_1\,\mathbf{n}_2] &= (\mathbf{t} \times \mathbf{m})\cdot(\mathbf{n}_1 \times \mathbf{n}_2)\\ &= \mathbf{t}\cdot\mathbf{n}_1\,\mathbf{m}\cdot\mathbf{n}_2 - \mathbf{t}\cdot\mathbf{n}_2\,\mathbf{m}\cdot\mathbf{n}_1\\ &= \Sigma\Sigma\,\varepsilon^{ij}\,\mathbf{t}\cdot\mathbf{n}_i\,\mathbf{m}\cdot\mathbf{n}_j.\end{aligned}$$

Differentiating $\mathbf{t}\cdot\mathbf{n} = 0$, $\mathbf{m}\cdot\mathbf{n} = 0$, and using 17.19, we see that

$$\begin{aligned}\mathbf{t}\cdot\mathbf{n}_i\,\mathbf{m}\cdot\mathbf{n}_j &= \mathbf{t}_i\cdot\mathbf{n}\,\mathbf{m}_j\cdot\mathbf{n} = \mathbf{t}_i\cdot\mathbf{n}\,\mathbf{n}\cdot\mathbf{m}_j\\ &= \mathbf{t}_i\cdot\mathbf{m}_j = (\mathbf{t}_i\cdot\mathbf{m})_j - \mathbf{t}_{ij}\cdot\mathbf{m}.\end{aligned}$$

Since $\Sigma\Sigma\,\varepsilon^{ij}\,\mathbf{t}_{ij} = 0$, it follows that

$$\begin{aligned}\sqrt{g}\,K &= \Sigma\Sigma\,\varepsilon^{ij}\,(\mathbf{t}_i\cdot\mathbf{m})_j = \Sigma\Sigma\,\varepsilon^{ij}\,(\mathbf{m}\cdot\mathbf{t}_i)_j\\ &= \Sigma\Sigma\,\varepsilon^{ij}\,[\mathbf{n}\,\mathbf{t}\,\mathbf{t}_i]_j.\end{aligned}$$

(We have interchanged the $\mathbf{t}$ and $\mathbf{m}$ of Kreyszig [**1**, p. 146].)

Since $\mathbf{n}_1 \times \mathbf{n}_2$ is parallel to $\mathbf{n}$, 19.48 may be expressed in the form

$$|\mathbf{n}_1 \times \mathbf{n}_2| = |K|\sqrt{g},$$

which can be used to establish Gauss's geometrical interpretation for K. To obtain his *spherical representation* of a surface, Gauss considered the locus of the end Q of a vector

$$\overrightarrow{OQ} = \mathbf{n},$$

where O is a fixed point and $\mathbf{n}$ is the unit normal at a point P which varies on the given surface [Hilbert and Cohn-Vossen **1**, pp. 193–196]. When P travels over a sufficiently small region F, bounded by a simple closed curve on the surface, Q travels over a corresponding region G of the unit sphere with center O. Gauss defined the *total curvature* of the surface at P to be the limit of the ratio of the areas of G and F when these regions are shrunk to single points. By 19.24, the area of F is

$$\iint|\mathbf{r}_1\,du^1 \times \mathbf{r}_2\,du^2| = \iint\sqrt{g}\,du^1\,du^2.$$

Analogously, the area of G is

$$\iint|\mathbf{n}_1\,du^1 \times \mathbf{n}_2\,du^2| = \iint|K|\sqrt{g}\,du^1\,du^2.$$

Hence the limit of the ratio is $|K|$.

The characteristic property of a developable surface is that, instead of a two-parameter family of tangent planes, it only has a one-parameter family of tangent planes, and so also a one-parameter family of normals. In this case G is not a proper region but merely an arc, and therefore $K = 0$.

If the parametric curves are orthogonal, so that $g_{12} = 0$, we have $g^{11} = 1/g_{11}$, $g^{12} = 0$, $g^{22} = 1/g_{22}$, whence, by 19.37,

$$b_i^j = \Sigma\,g^{jk}\,b_{ik} = g^{jj}\,b_{ij} = b_{ij}/g_{jj}.$$

In this case the mean curvature H is given by

$$2H = \Sigma\, b^j_j = \frac{b_{11}}{g_{11}} + \frac{b_{22}}{g_{22}}.$$

EXERCISES

1. Find the mean curvature H at a given point on the helicoid $y = x \tan(z/c)$, parametrized in the form

$$x = u^1 \cos u^2, \quad y = u^1 \sin u^2, \quad z = cu^2.$$

2. Find the mean curvature H at a given point on the general surface of revolution

$$\mathbf{r} = (u^1 \cos u^2, \quad u^1 \sin u^2, \quad z),$$

where z is a function of u^1 alone. This mean curvature is zero when z is given by

$$u^1 = a \cosh (z - c)/a,$$

i.e., when the surface is a catenoid.

3. Locate the curves of parabolic points on the torus

$$x = (a + b \cos u^1) \cos u^2, \quad y = (a + b \cos u^1) \sin u^2, \quad z = b \sin u^1.$$

4. The tangents $\mathbf{t} = \dot{\mathbf{r}}$ of a twisted curve $\mathbf{r} = \mathbf{r}(s)$ generate a surface

$$\mathbf{r}(s, u) = \mathbf{r}(s) + u\,\mathbf{t}(s).$$

Using s and u as parameters, obtain the fundamental magnitudes

$$b_{11} = \kappa\tau u, \qquad b_{12} = b_{22} = 0.$$

Deduce that $K = 0$ everywhere.

5. The mean curvature and Gauss curvature are connected by the inequality

$$H^2 \geqslant K.$$

At what kind of point do we find $H^2 = K$? (*Hint:* $H^2 - K = \frac{1}{4}(\kappa_{(1)} - \kappa_{(2)})^2$.)

6. Derive 19.48 another way, by applying Lagrange's identity to $(\mathbf{r}_1 \times \mathbf{r}_2)\cdot(\mathbf{n}_1 \times \mathbf{n}_2)$.

7. Derive 19.49 another way, by applying 17.62 in the form

$$\mathbf{t}_i = \nu_i \mathbf{m} - \mu_i \mathbf{n}, \quad \mathbf{m}_i = \lambda_i \mathbf{n} - \nu_i \mathbf{t}, \quad \mathbf{n}_i = \mu_i \mathbf{t} - \lambda_i \mathbf{m}$$

(where λ_i, μ_i, ν_i are functions of u^1 and u^2), so that

$$[\mathbf{n}\ \mathbf{n}_1\ \mathbf{n}_2] = \lambda_1\mu_2 - \lambda_2\mu_1 = \mathbf{m}_2\cdot\mathbf{t}_1 - \mathbf{m}_1\cdot\mathbf{t}_2.$$

19.5 PRINCIPAL DIRECTIONS AND LINES OF CURVATURE

Returning to 19.44, which may be expressed as

$$\Sigma\, b^k_j a^j = \kappa a^k \qquad (k = 1, 2),$$

we find that the easiest way to eliminate κ is to multiply by $\Sigma\, \varepsilon_{ik}\, a^i$ and sum over k, obtaining

$$\sum\sum\sum \varepsilon_{ik}\, b_j^k\, a^i\, a^j = \kappa \sum\sum \varepsilon_{ik}\, a^i\, a^k = 0.$$

(This sum is zero because the only nonvanishing terms involve $\varepsilon_{12}\, a^1\, a^2$ and $\varepsilon_{21}\, a^2\, a^1$, which cancel.) Thus the principal directions $\sum a^i\, \mathbf{r}_i$ are given by the roots of the quadratic equation

$$\sum\sum\sum \varepsilon_{ik}\, b_j^k\, a^i\, a^j = 0$$

for $a^1 : a^2$. In other words, the lines of curvature are determined by the differential equation

19.51 $$\sum\sum\sum \varepsilon_{ik}\, b_j^k\, du^i\, du^j = 0$$

or

19.52 $$-b_1^2\,(du^1)^2 + (b_1^1 - b_2^2)\, du^1\, du^2 + b_2^1\,(du^2)^2 = 0.$$

To prove that the lines of curvature form an *orthogonal* net, we may apply the criterion 19.291 either to 19.51 or to 19.52, using

$$c_{ij} = \sum \varepsilon_{ik}\, b_j^k.$$

We obtain, in the notation of 19.371,

$$\sum\sum g^{ij}\, c_{ij} = \sum\sum\sum \varepsilon_{ik}\, g^{ij}\, b_j^k = \sum\sum \varepsilon_{ik}\, b^{ik} = 0.$$

It follows that, at any point P on a surface, *the two principal directions are perpendicular.*

Another consequence of 19.44 is Rodrigues's formula

19.53 $$d\mathbf{n} + \kappa\, d\mathbf{r} = \mathbf{0},$$

which shows what happens to the normal $\mathbf{n}$ when $\mathbf{r}$ is displaced in a principal direction. (The coefficient κ is the corresponding principal curvature.) In fact, by combining the Weingarten equations

$$\mathbf{n}_i = -\sum b_i^j\, \mathbf{r}_j$$

(19.38) with 19.44 in the form

19.54 $$\sum b_i^j\, du^i = \kappa\, du^j,$$

we obtain

$$\begin{aligned} d\mathbf{n} &= \sum \mathbf{n}_i\, du^i = -\sum\sum b_i^j\, \mathbf{r}_j\, du^i \\ &= -\kappa \sum \mathbf{r}_j\, du^j = -\kappa\, d\mathbf{r}. \end{aligned}$$

(Olinde Rodrigues, 1794–1851.)

It follows that, if $d\mathbf{r}$ is in a principal direction, $d\mathbf{n}$ is in the same (or the opposite) direction. Moreover, the principal directions are the only directions in which this happens. For, if $d\mathbf{n}$ is parallel to $d\mathbf{r}$, the analysis given above shows that, for some number λ,

$$\sum\sum b_i^j\, \mathbf{r}_j\, du^i = \lambda \sum \mathbf{r}_j\, du^j.$$

Writing this in the form

$$\Sigma(\Sigma\, b_i^j\, du^i - \lambda\, du^j)\, \mathbf{r}_j = \mathbf{0},$$

we see that it implies

$$\Sigma\, b_i^j\, du^i - \lambda\, du^j = 0 \qquad (j = 1, 2).$$

Eliminating λ, the way we eliminated κ before, we again obtain 19.51, which is the differential equation for the lines of curvature.

If the parametric curves are orthogonal, so that $b_i^j = b_{ij}/g_{jj}$, the equation 19.52 becomes

19.55 $$-\frac{b_{12}}{g_{22}}(du^1)^2 + \left(\frac{b_{11}}{g_{11}} - \frac{b_{22}}{g_{22}}\right) du^1\, du^2 + \frac{b_{12}}{g_{11}}(du^2)^2 = 0.$$

For the investigation of a given surface, any net of curves on the surface may be used as parametric curves. The lines of curvature provide a standard net which is always available. Comparing 19.52 with $du^1\, du^2 = 0$, we see that

19.56 $$b_1^2 = b_2^1 = 0.$$

Hence

The parametric curves are the lines of curvature if and only if b_1^2 and b_2^1 are identically zero.

In this case the equation 19.45 reduces to

$$(\kappa - b_1^1)(\kappa - b_2^2) = 0,$$

so the two principal curvatures are b_1^1 and b_2^2. To see which is which, we apply Rodrigues's formula 19.53 to displacements along the parametric curves. The "first" principal direction is naturally the one along which u^1 varies while u^2 remains constant, that is, the direction of $\mathbf{r}_1$; and the "second" is the direction of $\mathbf{r}_2$. Thus

19.57 $$\mathbf{n}_i + \kappa_{(i)}\, \mathbf{r}_i = \mathbf{0}.$$

Taking inner products with $\mathbf{r}^j$ and $\mathbf{r}_j$, we deduce

$$-b_i^j + \kappa_{(i)}\, \delta_i^j = 0, \quad -b_{ij} + \kappa_{(i)}\, g_{ij} = 0.$$

Hence the two principal curvatures are precisely

19.58 $$\kappa_{(1)} = b_1^1 = \frac{b_{11}}{g_{11}}, \quad \kappa_{(2)} = b_2^2 = \frac{b_{22}}{g_{22}},$$

and we see also that

$$b_{12} = \kappa_{(1)}\, g_{12} = 0$$

(since the lines of curvature are orthogonal).

Conversely, if the parameters on any surface are so chosen that

19.59 $$g_{12} = 0, \qquad b_{12} = 0,$$

then the parametric curves are the lines of curvature; for, since

$$g^{21} = -\frac{g_{12}}{g} = 0,$$

we have

$$b_1^2 = \Sigma g^{2j} b_{1j} = g^{22} b_{12} = 0$$

and likewise $b_2^1 = 0$. In fact, the conditions 19.59 are equivalent to 19.56.

EXERCISES

1. Apply 19.291 to 19.52, so as to prove the orthogonality of the principal directions without using the symbol ε_{ik}.

2. The differential equation for the lines of curvature may be expressed in the form

$$\begin{vmatrix} g_{11} & g_{12} & g_{22} \\ b_{11} & b_{12} & b_{22} \\ (du^2)^2 & -du^1\,du^2 & (du^1)^2 \end{vmatrix} = 0.$$

3. Find the lines of curvatures on the hyperbolic paraboloid $x^2 - y^2 = 2z$, parametrized in the form

$$x = \sinh u^1 + \sinh u^2, \quad y = \sinh u^1 - \sinh u^2, \quad z = 2 \sinh u^1 \sinh u^2.$$

4. Find the lines of curvature on the helicoid $y = x \tan(z/c)$, parametrized in the form

$$x = u^1 \cos u^2, \quad y = u^1 \sin u^2, \quad z = cu^2.$$

5. The equations 19.56 or 19.59, holding at a particular point (but not necessarily identically), are conditions for the parametric directions to coincide with the principal directions at the point considered. The formulas 19.58 still hold at this point.

19.6 UMBILICS

An umbilic is a point at which the normal curvature κ is the same in all directions. At such a point, the equations 19.42, 19.43 are satisfied for all values of $a^1 : a^2$, and therefore

$$b_{ik} = \kappa g_{ik}, \qquad b_i^j = \kappa \delta_i^j.$$

In fact, we have two alternative sets of conditions for an umbilic: one set is

$$b_{11} : b_{12} : b_{22} = g_{11} : g_{12} : g_{22},$$

and the other,

19.61 $$b_1^2 = b_2^1 = 0, \qquad b_1^1 = b_2^2.$$

If a surface is symmetrical by reflection in a plane, its section by the plane is a line of curvature. To see why this happens, consider a point P on the

section. If the principal directions at P were oblique to the plane, either of them would reflect into another principal direction associated with the same principal curvature. Such an abundance of principal directions would make P an umbilic. But a curve consisting entirely of umbilics is, trivially, a line of curvature.

In particular, *on any surface of revolution, the meridians and parallels are lines of curvature.* An interesting special case arises when we rotate a plane curve about one of the normals of its evolute; that is, if C is the center of curvature for a point P on the plane curve, we rotate about the line through C parallel to the tangent at P. In this case the locus of P is an "equator" whose radius is equal to the radius of curvature of the meridian. The equator, like the meridian, is both a line of curvature and a normal section. Since the two principal curvatures are equal, every point on the equator is an umbilic.

At an umbilic, Rodrigues's formula 19.53 holds for all displacements on the surface. In particular, we can apply it to displacements along the parametric curves, obtaining

19.62 $$\mathbf{n}_j + \kappa\mathbf{r}_j = \mathbf{0} \qquad (j = 1, 2).$$

Hence

19.63 *If every point is an umbilic, the surface is either a plane or a sphere.*

For, in this case 19.62 holds everywhere. Differentiating, we deduce

$$\mathbf{n}_{ij} + \kappa\mathbf{r}_{ij} + \kappa_i \mathbf{r}_j = \mathbf{0}, \qquad \kappa_1 \mathbf{r}_2 = \kappa_2 \mathbf{r}_1,$$

and $\kappa_1 = \kappa_2 = 0$. Thus κ is constant, and 19.62 yields $(\mathbf{n} + \kappa\mathbf{r})_j = \mathbf{0}$, so that $\mathbf{n} + \kappa\mathbf{r}$ is constant. If $\kappa = 0$, $\mathbf{n}$ is constant, and we have a plane. If $\kappa \neq 0$, a suitable origin makes $\mathbf{r} = -\kappa^{-1}\mathbf{n}$, $|\mathbf{r}| = |\kappa|^{-1}$, and we have a sphere.

EXERCISES

1. How can the equation 19.52 be used to derive the conditions

$$b_1^2 = b_2^1 = 0, \qquad b_1^1 = b_2^2$$

for an umbilic?

2. What happens to the equation 19.45 when these conditions are satisfied?
3. Anticlastic surfaces have no umbilics.
4. The surface

$$x = \sqrt{2}\cos u^1, \quad y = \sqrt{2}\cos u^2, \quad z = \sin u^1 \sin u^2$$

has a curve of umbilics lying on the sphere $x^2 + y^2 + z^2 = 4$.

5. Does every umbilic lie on infinitely many lines of curvature?

19.7 DUPIN'S THEOREM AND LIOUVILLE'S THEOREM

Dupin investigated triply orthogonal families of surfaces, not as a barren exercise in the differential calculus but because certain instances of such families are of the first importance in . . . mathematical physics. [They] were the occasion for one of Darboux' more famous works, extending to 567 pages.

E. T. Bell [**2**, p.331]

In the exercises at the end of § 18.5, we found several instances of three-dimensional coordinates having the special property

$$g_{23} = g_{31} = g_{12} = 0,$$

so that the level surfaces all cut one another at right angles. In such a case the three systems of surfaces are said to be *mutually orthogonal.*

Differentiating the equation $\mathbf{r}_1 \cdot \mathbf{r}_2 = 0$ with respect to u^3, we obtain

$$\mathbf{r}_{13} \cdot \mathbf{r}_2 + \mathbf{r}_1 \cdot \mathbf{r}_{23} = 0.$$

From this and two other such equations (derived by cyclic permutation of 123) we deduce

$$\mathbf{r}_1 \cdot \mathbf{r}_{23} = \mathbf{r}_2 \cdot \mathbf{r}_{31} = \mathbf{r}_3 \cdot \mathbf{r}_{12} = 0.$$

Since $\mathbf{r}_3$ is normal to the surface $u^3 = c$ of the third system, this surface satisfies not only $g_{12} = 0$ but also, since $\mathbf{r}_3 \cdot \mathbf{r}_{12} = 0$,

$$b_{12} = \mathbf{n} \cdot \mathbf{r}_{12} = 0,$$

as in 19.59. Hence the parametric curves $u^1 = a$ and $u^2 = b$ on this surface are lines of curvature. Since a similar result holds for a surface of either of the other systems, we have now proved

DUPIN'S THEOREM. *In three mutually orthogonal systems of surfaces, the lines of curvature on any surface in one of the systems are its intersections with the surfaces of the other two systems.*

Moreover, any surface may be exhibited as a member of one of three mutually orthogonal systems. This can actually be done in many ways. One way is to use a system of *parallel* surfaces, defined as the loci of points at constant distances along the normals of the given surface. Using the lines of curvature as parametric curves, we may express the position vector of a typical parallel surface in the form

$$\bar{\mathbf{r}} = \mathbf{r} + u^3\,\mathbf{n}.$$

We see from 19.57 that the directions of the parametric curves on the new surface are given by

$$\begin{aligned}\bar{\mathbf{r}}_i &= (\mathbf{r} + u^3\mathbf{n})_i = \mathbf{r}_i + u^3\mathbf{n}_i \\ &= \mathbf{r}_i - u^3\kappa_{(i)}\,\mathbf{r}_i = (1 - \kappa_{(i)}\,u^3)\,\mathbf{r}_i,\end{aligned}$$

that is, they are parallel to those on the original surface. Therefore, both surfaces have the same normal: $\bar{\mathbf{n}} = \mathbf{n}$. Since

$$\bar{g}_{12} = \bar{\mathbf{r}}_1 \cdot \bar{\mathbf{r}}_2 = 0$$

and

$$\bar{b}_{12} = -\bar{\mathbf{n}}_1 \cdot \bar{\mathbf{r}}_2 = -\mathbf{n}_1 \cdot \bar{\mathbf{r}}_2 = \kappa_{(1)} \mathbf{r}_1 \cdot \bar{\mathbf{r}}_2 = 0,$$

the parametric curves on the new surface are again lines of curvature [Weatherburn **2**, p. 159]. Allowing u^3 to take various values, we obtain a whole system of parallel surfaces. The other two orthogonal systems are traced out by the normals at points that run along the lines of curvature [La Vallée Poussin **2**, p. 447].

In § 6.7 and § 7.7 we analysed the general circle-preserving and sphere-preserving transformations. In § 9.7 we indicated how the theory of functions of a complex variable could be employed to prove that the circle-preserving transformations are the only angle-preserving transformations of the whole inversive plane into itself. It is very remarkable that the analogous theorem in three (or more) dimensions is elementary! We merely have to observe that, if a transformation of space preserves the angles at which surfaces cut, it transforms mutually orthogonal systems of surfaces into mutually orthogonal systems of surfaces. Hence, if it transforms a surface σ into another surface σ', it transforms the lines of curvature on σ into the lines of curvature on σ'. Since a sphere (including a plane as a special case) is characterized by the property that *all* directions on it are principal directions, we can immediately deduce

LIOUVILLE'S THEOREM. *Every angle-preserving transformation is a sphere-preserving transformation.*

Taking this along with 7.71, we see that every angle-preserving transformation is either a similarity or the product of an inversion and an isometry [Forder **3**, pp. 137–138].

EXERCISES

1. When the surfaces $u^1 = a$ and $u^2 = b$ are traced out by normals along lines of curvature, while the surfaces $u^3 = c$ are "parallel," the fundamental magnitudes for $u^1 = a$ are naturally denoted by $g_{22}, g_{23}, g_{33}, b_{22}, b_{23}, b_{33}$. Dupin's theorem shows that $g_{23} = b_{23} = 0$. Compute b_{33}, and deduce that for this surface $K = 0$.

2. The central quadrics

$$\frac{x^2}{A-\lambda} + \frac{y^2}{B-\lambda} + \frac{z^2}{C-\lambda} = 1 \qquad (A > B > C)$$

(which are ellipsoids when $\lambda < C$, hyperboloids of one sheet when $C < \lambda < B$, hyperboloids of two sheets when $B < \lambda < A$) are said to be *confocal*. At a point (x, y, z) on such a quadric, the direction of the normal is

$$\left(\frac{x}{A-\lambda}, \frac{y}{B-\lambda}, \frac{z}{C-\lambda} \right).$$

When x, y, z are given, 18.59 is a cubic equation for λ, whose roots u^1, u^2, u^3 (numbered in increasing order) satisfy

$$u^1 < C < u^2 < B < u^3 < A.$$

Deduce that any point for which $xyz \neq 0$ lies on three quadrics of the system (one of each kind), cutting one another orthogonally [La Vallée Poussin **2**, p. 448].

3. Where are the lines of curvature on an ellipsoid? [Hilbert and Cohn-Vossen **1**, p. 189.]

19.8 DUPIN'S INDICATRIX

Although the indicatrix was not invented by Dupin, he made more effective use than had his predecessors of this suggestive conic in which a plane parallel to, and "infinitesimally near to," the tangent plane at any point of a surface intersects the surface.

E. T. Bell [**2**, p. 331]

When a surface is given in Monge's form $z = F(x, y)$, it is often convenient to use the coordinates x, y themselves as parameters, so that z is a function of x and y with derivatives

$$z_1 = \frac{\partial z}{\partial x}, \quad z_2 = \frac{\partial z}{\partial y}, \quad z_{11} = \frac{\partial^2 z}{\partial x^2}, \quad z_{12} = \frac{\partial^2 z}{\partial x\, \partial y}, \quad z_{22} = \frac{\partial^2 z}{\partial y^2}.$$

Differentiating $\mathbf{r} = (x, y, z)$, we obtain

$$\mathbf{r}_1 = (1, 0, z_1), \quad \mathbf{r}_2 = (0, 1, z_2), \quad \mathbf{r}_{ij} = (0, 0, z_{ij}),$$

whence

$$g_{11} = 1 + z_1{}^2, \quad g_{12} = z_1 z_2, \quad g_{22} = 1 + z_2{}^2,$$

$$g = g_{11}\, g_{22} - g_{12}{}^2 = 1 + z_1{}^2 + z_2{}^2,$$

$$\sqrt{g}\, \mathbf{n} = \mathbf{r}_1 \times \mathbf{r}_2 = (-z_1, -z_2, 1),$$

$$\sqrt{g}\, b_{ij} = \sqrt{g}\, \mathbf{n} \cdot \mathbf{r}_{ij} = z_{ij}.$$

If the coordinate axes are so chosen that the point under consideration is the origin and the normal there is the z-axis, we have

$$\mathbf{n} = (0, 0, 1),$$

whence $\quad z_1 = z_2 = 0, \quad g_{11} = 1, \quad g_{12} = 0, \quad g_{22} = 1, \quad g = 1, \quad b_{ij} = z_{ij}.$

Since z is a function of x and y, we can expand it in a Maclaurin series

$$\begin{aligned} z &= z(0,0) + z_1 x + z_2 y + \tfrac{1}{2}(z_{11} x^2 + 2z_{12} xy + z_{22} y^2) + \tfrac{1}{6}(z_{111} x^3 + \dots) + \dots \\ &= \tfrac{1}{2}(b_{11} x^2 + 2b_{12} xy + b_{22} y^2) + \text{(terms of higher degree in } x \text{ and } y). \end{aligned}$$

The "terms of higher degree" become important if $b_{11} = b_{12} = b_{22} = 0$, in which case the origin is a parabolic umbilic. In all other cases the section by a plane $z = \varepsilon$, parallel to the tangent plane $z = 0$ at a small distance $|\varepsilon|$, resembles the conic

$$b_{11} x^2 + 2b_{12} xy + b_{22} y^2 = 2\varepsilon,$$

which is similar to *Dupin's indicatrix*

19.81 $$b_{11} x^2 + 2b_{12} xy + b_{22} y^2 = \pm 1.$$

We shall find that this conic (or pair of conics) indicates, in a remarkably simple manner, the normal curvature in every direction. We first observe that this part of the surface is synclastic, anticlastic, or parabolic, according as

$$b > 0, \quad b < 0, \quad \text{or} \quad b = 0,$$

that is, according as the indicatrix is an ellipse, two conjugate hyperbolas, or two parallel lines. (In the synclastic case the ambiguous sign in the equation 19.81 must agree with the sign of b_{11}; otherwise the plane $z = \varepsilon$ would fail to meet the surface. In the anticlastic case we need both signs: one for each of the two conjugate hyperbolas.)

In the plane $z = 0$, which is the tangent plane at the origin, the vector $(\cos\theta, \sin\theta, 0)$, making an angle θ with the x-axis $(1, 0, 0)$, may be identified with the tangent

$$\overrightarrow{PT} = \mathbf{t} = \Sigma\, a^i\, \mathbf{r}_i = (a^1, a^2, 0)$$

of Figure 19.2*a*. Thus the contravariant components of $\mathbf{t}$ are $a^1 = \cos\theta$, $a^2 = \sin\theta$, and, by 19.35, the normal curvature in this direction is

$$\kappa = \Sigma\Sigma\, b_{ij}\, a^i\, a^j$$

19.82 $$= b_{11}\cos^2\theta + 2b_{12}\cos\theta\sin\theta + b_{22}\sin^2\theta.$$

Expressing the indicatrix 19.81 in polar coordinates, we obtain

$$b_{11} r^2 \cos^2\theta + 2b_{12} r^2 \cos\theta\sin\theta + b_{22} r^2 \sin^2\theta = \pm 1,$$

that is, $\kappa r^2 = \pm 1$, or

$$r = |\kappa|^{-\frac{1}{2}}.$$

In other words [La Vallée Poussin **2,** p. 427],

The radius of the indicatrix in any direction is equal to the square root of the radius of normal curvature in this direction.

Another way of expressing the same idea is to remark that the surface is approximated by the paraboloid or parabolic cylinder

$$2z = b_{11} x^2 + 2b_{12} xy + b_{22} y^2.$$

In any direction (at the origin) the given surface and the quadric have the same normal curvature.

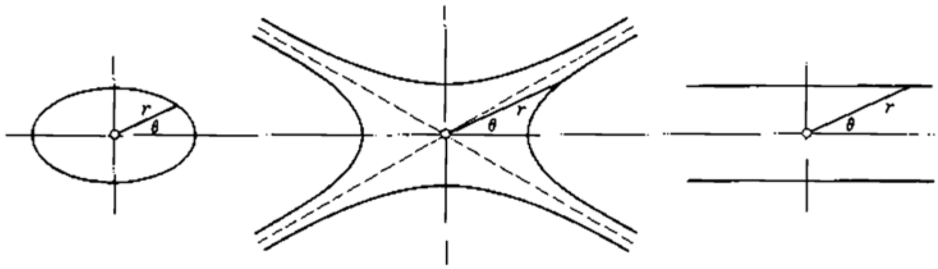


Figure 19.8a

If we choose the x- and y-axes along the principal directions at the origin, as in Figure 19.8a, so that $b_{12} = 0$ and, by 19.58, $\kappa_{(i)} = b_{ii}$, the indicatrix 19.81 is simply

$$\kappa_{(1)} x^2 + \kappa_{(2)} y^2 = \pm 1,$$

and 19.82 yields *Euler's formula*

$$\kappa = \kappa_{(1)} \cos^2 \theta + \kappa_{(2)} \sin^2 \theta$$

for the normal curvature in a direction making an angle θ with the first principal direction.

EXERCISES

1. For which directions on a surface is the normal curvature equal to the arithmetic mean of the two principal curvatures?

2. For the surface $z = F(x, y)$,

$$2H = \frac{g_{22} b_{11} - 2g_{12} b_{12} + g_{11} b_{22}}{g} = \frac{(1 + z_2^2)z_{11} - 2z_1 z_2 z_{12} + (1 + z_1^2)z_{22}}{(1 + z_1^2 + z_2^2)^{3/2}} .$$

3. The surface $xyz = 1$ has umbilics at the four vertices of a regular tetrahedron [Salmon **2**, p. 300].

4. The surface $z = x(x^2 - 3y^2)$ has a parabolic umbilic at the origin. Sketch the section by a plane $z = \epsilon$, where ϵ is a small number, positive or negative.

This surface is called the *monkey saddle* because it would be the right kind of saddle for a monkey riding a bicycle: one way down for each hind leg and a third for the tail. Hilbert and Cohn-Vossen [**1**, p. 202, Fig. 213] made a nice drawing of the surface but a very misleading one of its generalized indicatrix [*ibid.*, p. 192, Fig. 200]. For the true shape, see the second edition of Struik [**1**, p. 85].

20

Geodesics

Imagine a two-dimensional creature, sufficiently intelligent to make precise measurements on the surface he inhabited, but unable to conceive of a third dimension. His world might be a plane or a parabolic cylinder; he could not tell the difference. If it were a circular cylinder, local measurements would still give the same results, though an expedition all the way round would reveal a topological peculiarity. In all these cases his conclusion would be that his surface had zero curvature: $K = 0$. If, on the other hand, the surface were a sphere, he could detect its positive curvature by measurements within easy reach of his home, even if the radius of the sphere were relatively large. This information is a consequence of Gauss's formula 20.16, which expresses K in terms of the fundamental magnitudes of the first order. In § 20.3 we shall see how Gauss's complicated expression can be replaced by his simple one involving the three angles of a triangle (like our formula 6.92 for the area of a spherical triangle). In § 20.4 we shall extend these local measurements to global measurements, which would enable our intelligent ant to determine the topological nature of his world: an idea which we shall investigate more systematically in Chapter 21.

The remaining sections of the present chapter deal with the differential-geometric approach to the non-Euclidean planes, which may be regarded as surfaces of constant curvature.

20.1 THEOREMA EGREGIUM

The Christoffel symbols are called after Erwin Bruno Christoffel (1829 - 1901), who introduced [them] in 1869, . . . denoting our Γ^i_{jk} by $\{^{jk}_{\,i}\}$. The change to our present notation has been made under the influence of tensor theory.

D. J. Struik (1894 -   )

[Struik **1**, p. 108]

For an adequate discussion of "geodesics," which are the most important

curves on a surface, we need one more notational device: the *Christoffel symbols*

20.11 $$\Gamma_{ij,k} = \mathbf{r}_{ij}\cdot\mathbf{r}_k, \quad \Gamma_{ij}^k = \mathbf{r}_{ij}\cdot\mathbf{r}^k.$$

In virtue of 19.15, these symbols are related as follows:

20.12 $$\Gamma_{ij,k} = \Sigma g_{kl}\,\Gamma_{ij}^l, \quad \Gamma_{ij}^k = \Sigma g^{kl}\,\Gamma_{ij,l}$$

Clearly, $\Gamma_{21,k} = \Gamma_{12,k}$ and $\Gamma_{21}^k = \Gamma_{12}^k$.

Since the derivatives of the fundamental magnitudes g_{ij} are

$$(g_{ij})_k = \frac{\partial}{\partial u^k}(\mathbf{r}_i\cdot\mathbf{r}_j) = \mathbf{r}_{ik}\cdot\mathbf{r}_j + \mathbf{r}_i\cdot\mathbf{r}_{jk}$$

20.13 $$= \Gamma_{ik,j} + \Gamma_{jk,i},$$

we can compute the Christoffel symbols of the first kind by means of the formula

20.14 $$\Gamma_{ij,k} = \tfrac{1}{2}\{(g_{jk})_i + (g_{ik})_j - (g_{ij})_k\}$$

and then deduce the Christoffel symbols of the second kind by means of the latter half of 20.12.

Applying 18.21 with $\mathbf{u} = \mathbf{r}_{ij}$ and $\mathbf{r}_3 = \mathbf{r}^3 = \mathbf{n}$, we obtain

$$\mathbf{r}_{ij} = \mathbf{r}_{ij}\cdot\mathbf{r}^1\,\mathbf{r}_1 + \mathbf{r}_{ij}\cdot\mathbf{r}^2\,\mathbf{r}_2 + \mathbf{r}_{ij}\cdot\mathbf{r}^3\,\mathbf{r}_3$$

$$= \Gamma_{ij}^1\mathbf{r}_1 + \Gamma_{ij}^2\mathbf{r}_2 + \mathbf{r}_{ij}\cdot\mathbf{n}\,\mathbf{n}$$

20.15 $$= \Sigma\,\Gamma_{ij}^k\,\mathbf{r}_k + b_{ij}\,\mathbf{n}.$$

These expressions for the second derivatives of $\mathbf{r}$ are known as "the equations of Gauss."

Another of Gauss's discoveries, which pleased him so much that he named it *Theorema egregium,* is an expression for K in terms of the g's and their derivatives. This means that the Gaussian curvature can be computed by measurements made on the surface itself, without reference to the three-dimensional space in which it may lie. In other words, K is a "bending invariant," unchanged by the kind of distortion that takes place when a flat sheet of paper is rolled up to make a cylinder or a cone. The expression appears in the literature in various forms, not obviously identical. One of the neatest, discovered by Liouville,* is

20.16 $$\sqrt{g}\,K = \frac{\partial}{\partial u^2}\left(\frac{\sqrt{g}}{g_{11}}\Gamma_{11}^2\right) - \frac{\partial}{\partial u^1}\left(\frac{\sqrt{g}}{g_{11}}\Gamma_{12}^2\right).$$

This can be derived by applying 19.49 to the unit tangent vector $\mathbf{t} = \mathbf{r}_1/g_1$. Since $[\mathbf{n}\;\mathbf{r}_1\;\mathbf{r}_1] = 0$ and (by 19.33) $\mathbf{n}\times\mathbf{r}_1 = \sqrt{g}\,\mathbf{r}^2$, we have

* J. Liouville, Sur la théorie générale des surfaces, *Journal de Mathématiques* **16** (1851), pp. 130–132.

$$[\mathbf{n}\,\mathbf{t}\,\mathbf{t}_i] = \left[\mathbf{n}\,\frac{\mathbf{r}_1}{g_1}\,\frac{\mathbf{r}_{1i}}{g_1}\right] = (\mathbf{n}\times\mathbf{r}_1)\cdot\frac{\mathbf{r}_{1i}}{g_{11}}$$

$$= \sqrt{g}\,\mathbf{r}^2\cdot\frac{\mathbf{r}_{1i}}{g_{11}} = \frac{\sqrt{g}}{g_{11}}\,\Gamma^2_{1i}.$$

Hence
$$\sqrt{g}\,K = \Sigma\Sigma\varepsilon^{ij}\left(\frac{\sqrt{g}}{g_{11}}\,\Gamma^2_{1i}\right)_j$$

$$= \left(\frac{\sqrt{g}}{g_{11}}\,\Gamma^2_{11}\right)_2 - \left(\frac{\sqrt{g}}{g_{11}}\,\Gamma^2_{12}\right)_1.$$

Theorema egregium takes a more symmetrical form when the parametric curves are orthogonal, so that

$$g_{12} = 0, \quad g^{ii} = 1/g_{ii} = 1/(g_i)^2,$$

$$\frac{\sqrt{g}}{g_{11}}\,\Gamma^2_{12} = \frac{g_1 g_2}{g_{11}}\,\frac{(g_{22})_1}{2g_{22}} = \frac{g_2}{g_1}\,\frac{2g_2(g_2)_1}{2g_2{}^2} = \frac{(g_2)_1}{g_1}$$

and

$$\frac{\sqrt{g}}{g_{11}}\,\Gamma^2_{11} = \frac{g_1 g_2}{g_{11}}\left(-\frac{(g_{11})_2}{2g_{22}}\right) = -\frac{g_2}{g_1}\,\frac{2g_1(g_1)_2}{2g_2{}^2} = -\frac{(g_1)_2}{g_2}.$$

In fact,

$$-g_1 g_2 K = \left(\frac{\sqrt{g}}{g_{11}}\,\Gamma^2_{12}\right)_1 - \left(\frac{\sqrt{g}}{g_{11}}\,\Gamma^2_{11}\right)_2$$

20.17
$$= \left(\frac{(g_2)_1}{g_1}\right)_1 + \left(\frac{(g_1)_2}{g_2}\right)_2$$

[Weatherburn **2**, p. 98; Struik **1**, p. 113].

EXERCISES

1. Obtain a variant of 20.16 by taking $\mathbf{t} = \mathbf{r}_2/g_2$.

2. For the case when $g_{12} = 0$, express all the Christoffel symbols in terms of g_{11}, g_{22}, and their derivatives.

3. Compute the Christoffel symbols for polar coordinates in the plane.

4. For a surface in Monge's form $z = F(x, y)$,

$$\Gamma^k_{ij} = z_{ij}\,z_k/(1 + z_1{}^2 + z_2{}^2).$$

5. $\Sigma\,\Gamma^i_{ij} = (\log\sqrt{g})_j$.

6. *Compute K for a surface on which

$$g_{11} = \frac{(u^2)^2 + a^2}{\{(u^1)^2 + (u^2)^2 + a^2\}^2}, \quad g_{12} = \frac{-u^1 u^2}{\{(u^1)^2 + (u^2)^2 + a^2\}^2},$$

$$g_{22} = \frac{(u^1)^2 + a^2}{\{(u^1)^2 + (u^2)^2 + a^2\}^2}.$$

* E. Beltrami, *Annali di Matematica* (1) **7** (1866), pp. 197–198.

20.2 THE DIFFERENTIAL EQUATIONS FOR GEODESICS

Every sufficiently small portion of a geodesic is the shortest path on the surface connecting the end-points of the portion. . . . All the intrinsic properties of a surface (such as its Gaussian curvature) can be determined by drawing geodesics and measuring their arc lengths. . . . We can obtain an approximation to the geodesics by moving a very small buggy along the surface on two wheels, the wheels being rigidly fastened to their common axis so that their speeds of rotation are equal. . . . The entire course of a geodesic is determined if one of its points and its direction at this point are given. . . . The straightest lines may also be characterized by the geometric requirement that the osculating plane of the curve is to contain the normal to the surface at every point of the curve.

Hilbert and Cohn-Vossen
[**1**, pp. 220-221]

Consider the possibility of a curve on a surface having all its principal normals normal to the surface. As we saw in § 19.3, any curve on a surface satisfies

$$\kappa\mathbf{p} = \dot{\mathbf{t}} = \frac{d}{ds}\,\Sigma\,\dot{u}^i\,\mathbf{r}_i = \Sigma\,\ddot{u}^i\,\mathbf{r}_i + \Sigma\Sigma\,\dot{u}^i\,\dot{u}^j\,\mathbf{r}_{ij}.$$

In the present case, since $\mathbf{p} = \mathbf{n}$ is perpendicular to $\mathbf{r}^k$, we have

$$(\Sigma\,\ddot{u}^i\,\mathbf{r}_i + \Sigma\Sigma\,\dot{u}^i\,\dot{u}^j\,\mathbf{r}_{ij})\cdot\mathbf{r}^k = 0.$$

Since $\mathbf{r}_i\cdot\mathbf{r}^k = \delta_i^k$ and $\mathbf{r}_{ij}\cdot\mathbf{r}^k = \Gamma_{ij}^k$, these equations reduce to

20.21 $$\ddot{u}^k + \Sigma\Sigma\,\Gamma_{ij}^k\,\dot{u}^i\,\dot{u}^j = 0 \qquad (k = 1, 2),$$

meaning

$$\frac{d^2u^k}{ds^2} + \Sigma\Sigma\,\Gamma_{ij}^k\,\frac{du^i}{ds}\frac{du^j}{ds} = 0$$

[Struik **1**, p. 132]. Theoretically, we could eliminate s from these two equations so as to obtain a single differential equation; but it is usually more convenient either to use both equations or to use one of them along with 19.16.

The curves so determined are called *geodesics* [Weatherburn **2**, p. 100]. Since the equations express the second derivatives of u^k in terms of the first derivatives, there is a geodesic through any given point A (on the surface) in any given direction. There is also, in general, a unique geodesic joining two given points A and B. In these respects the geodesics on a surface resemble the straight lines in a plane; in fact, as we shall see, they are straight when the surface is a plane.

When $g_{12} = 0$, so that the parametric curves are orthogonal, the differential equations take the form

20.22 $$g_{11}\ddot{u}^1 + \tfrac{1}{2}(g_{11})_1(\dot{u}^1)^2 + (g_{11})_2\dot{u}^1\dot{u}^2 - \tfrac{1}{2}(g_{22})_1(\dot{u}^2)^2 = 0,$$

20.23 $$g_{22}\ddot{u}^2 - \tfrac{1}{2}(g_{11})_2(\dot{u}^1)^2 + (g_{22})_1\dot{u}^1\dot{u}^2 + \tfrac{1}{2}(g_{22})_2(\dot{u}^2)^2 = 0.$$

The latter shows that the parametric curves $\dot{u}^2 = 0$ occur among the geodesics if and only if g_{11} is a function of u^1 alone (not involving u^2), so that $(g_{11})_2 = 0$. In this case the curves $\dot{u}^2 = 0$ are a one-parameter system of geodesics and the curves $\dot{u}^1 = 0$ are their orthogonal trajectories. Since g_{11} is a function of u^1 alone, the differential form

$$ds^2 = g_{11}(du^1)^2 + g_{22}(du^2)^2$$

can be simplified by changing the notation so that $\int g_1\, du^1$ is called u^1. Then $g_{11} = 1, \quad g = g_{22}, \quad$ and

20.24 $$ds^2 = (du^1)^2 + g(du^2)^2.$$

The effect of this change of notation is to make u^1 measure the arc of each geodesic $u^2 =$ constant. The differential equations are now

20.25 $$\ddot{u}^1 - \tfrac{1}{2}(g)_1(\dot{u}^2)^2 = 0,$$

20.26 $$\frac{d}{ds}(g\dot{u}^2) - \tfrac{1}{2}(g)_2(\dot{u}^2)^2 = 0.$$

In particular, we obtain *geodesic polar* coordinates (analogous to ordinary polar coordinates in the plane) by measuring u^1 from A along all the geodesics through A, and defining u^2 to be the angle that such a geodesic $u^2 =$ constant makes with an "initial" geodesic $u^2 = 0$.

The length of any curve from A to a point B (of general position) is obtained by integrating ds along the curve. The equation 20.24 shows that $\int ds \geqslant \int du^1$, with equality only when $du^2 = 0$. Hence the geodesic AB is the *shortest path* from A to B. In fact, it is the curve along which a tightly stretched string would lie on the smooth convex side of a material surface. Since the only forces acting on an "element" of the string are the tensions at its two ends and the reaction of the surface along the normal $\mathbf{n}$, these three forces must be in equilibrium. Hence $\mathbf{n}$ must lie in the plane of the two tensions, which is the osculating plane of the curve. These considerations provide a statical explanation for the equation $\mathbf{p} = \mathbf{n}$ which started this investigation.

The curves $u^1 =$ constant (which, of course, are not geodesics) are called "geodesic circles." The circumference of such a "circle" is obtained by integrating ds (given by 20.24 with $du^1 = 0$); thus it is

$$\int_0^{2\pi} \sqrt{g}\, du^2.$$

When the radius u^1 is small, this circumference is approximated by both $2\pi\sqrt{g}$ and $2\pi u^1$. Hence the first term in the Maclaurin series for $\sqrt{g}$ is simply u^1, that is,

20.27 If $u^1 = 0$, then $(\sqrt{g})_1 = 1$.

Since geodesics are curves of shortest length, the geodesics on a sphere are the great circles, and the geodesics in a plane are the straight lines. We can adapt Figure 8.5*b* to geodesic polar coordinates by writing u^1 and u^2 for r and θ, so that 20.24 expresses Pythagoras's theorem for the infinitesimal triangle $PP'N$, and the angle ϕ between the geodesics OP' and PP' is given by

$$\cos\phi = \lim \frac{NP'}{PP'} = \dot{u}^1 \quad \text{or} \quad \sin\phi = \lim \frac{NP}{PP'} = \sqrt{g}\,\dot{u}^2.$$

Differentiating $\cos\phi$ and using 20.25, we obtain

$$-\sin\phi\,\dot{\phi} = \ddot{u}^1 = \tfrac{1}{2}(g)_1(\dot{u}^2)^2 = \frac{(g)_1}{2\sqrt{g}}(\sqrt{g}\,\dot{u}^2)\dot{u}^2$$
$$= (\sqrt{g})_1(\sin\phi)\dot{u}^2.$$

Thus

20.28 $$d\phi = -(\sqrt{g})_1\,du^2.$$

EXERCISES

1. In the case of the general surface of revolution

$$\mathbf{r} = (u^1 \cos u^2, \quad u^1 \sin u^2, \quad z),$$

where z is a function of u^1 alone, the differential equation 20.23 for geodesics reduces to

$$\frac{d}{ds}\left[(u^1)^2\,\dot{u}^2\right] = 0.$$

One solution is $du^2 = 0$, showing that *the meridians are geodesics*. In all other cases the constant value of $(u^1)^2\,\dot{u}^2$ may be denoted by $1/h$, so that

$$ds = h\,(u^1)^2\,du^2.$$

Comparing this with 19.16, obtain the complete integral

$$u^2 = C \pm \int \left[\frac{1 + z_1{}^2}{h^2(u^1)^2 - 1}\right]^{\frac{1}{2}} \frac{du^1}{u^1}$$

[Weatherburn **2**, p. 102].

2. The geodesics on a cylinder are helices.
3. The geodesics on the cone

$$\mathbf{r} = (u^1 \cos u^2, \quad u^1 \sin u^2, \quad u^1 \cos\alpha)$$

are given by

$$au^1 = \sec(\beta + u^2 \sin\alpha),$$

where α and β are constants. Are these curves concho-spirals?

20.3 THE INTEGRAL CURVATURE OF A GEODESIC TRIANGLE

The integral curvature of a region on the surface is equal to the area of its spherical image. . . . This property of the integral curvature was already known to the French school of Monge before Gauss pointed out its significance for the intrinsic geometry of a surface.

D. J. Struik [**1**, p. 156]

Formulas 6.92 and 16.53, for the areas of spherical and hyperbolic triangles, are special cases of a beautiful formula which Gauss discovered for the *integral curvature* of a triangle formed by arcs of three geodesics on any smooth surface. We proceed to establish this formula

20.31
$$\iint_{ABC} K\,dS = A + B + C - \pi.$$

Setting $g_1 = 1$ and $g_2 = \sqrt{g}$ in 20.17, we see that, when geodesic polar coordinates are used, the formula for K is simply

20.32
$$\sqrt{g}\,K = -(\sqrt{g})_{11}.$$

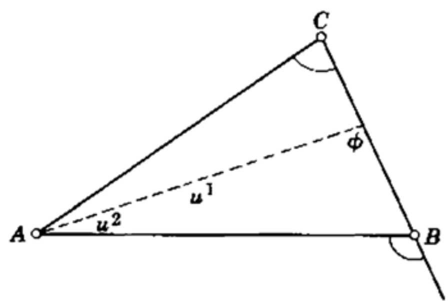


Figure 20.3a

Consider a geodesic triangle ABC with its side AB along the initial geodesic $u^2 = 0$, as in Figure 20.3*a*. Integrating K over the area of this triangle with the help of 19.24, we obtain

$$\begin{aligned}\iint K\,dS &= \iint K\sqrt{g}\,du^1\,du^2 = -\iint(\sqrt{g})_{11}\,du^1\,du^2\\ &= -\int (\sqrt{g})_1\Big|_0^{u^1}\,du^2.\end{aligned}$$

By 20.27, $(\sqrt{g})_1 = 1$ when $u^1 = 0$; and by 20.28, $-(\sqrt{g})_1\,du^2 = d\phi$ for any point on the geodesic BC. Hence

$$\begin{aligned}\iint_{ABC} K\,dS &= \int_0^A \{1 - (\sqrt{g})_1\}\,du^2 = \int_0^A du^2 + \int_{\pi-B}^{C} d\phi\\ &= A + C - (\pi - B) = A + B + C - \pi\end{aligned}$$

[Weatherburn **2**, p. 117].

EXERCISES

1. Obtain 20.32 directly from 19.49 with $\mathbf{t} = \mathbf{r}_1$.

2. On the unit sphere, colatitude and longitude serve as geodesic polar coordinates with $g = \sin^2 u^1$. What happens to 20.32 in this case? For convenience, write r, θ for u^1, u^2. The differential equation 20.26 (with $(g)_2 = 0$) has, as a first integral,

$$(\sin^2 r)\dot{\theta} = 1/h$$

(an arbitrary constant). Combining this formula for $\dot{\theta} = d\theta/ds$ with

$$ds^2 = dr^2 + \sin^2 r\, d\theta^2,$$

deduce $dr = \sin r\, \sqrt{h^2 \sin^2 r - 1}\, d\theta$,

$$\theta = \int \frac{\csc^2 r\, dr}{\sqrt{h^2 - \csc^2 r}} = \theta_0 + \arccos(k \cot r)$$

where $k = 1/\sqrt{h^2 - 1}$, and

$$k \cot r = \cos(\theta - \theta_0).$$

Expressing this solution in terms of the Cartesian coordinates

$$x = \sin r \cos \theta, \quad y = \sin r \sin \theta, \quad z = \cos r,$$

verify that the geodesics on a sphere are the great circles (lying in planes through the origin).

20.4 THE EULER-POINCARÉ CHARACTERISTIC

As we saw on page 281, any polyhedron inscribed in a sphere can be projected from the center onto the surface of the sphere so as to form a *map*. In fact, the V vertices are joined in pairs by E geodesic arcs (which we still call edges), decomposing the spherical surface into F polygonal regions (which we still call faces). More generally, a map may be obtained by drawing a sufficient number of geodesic arcs on any closed surface. We can insist that the points ("vertices") be so placed and so joined that every face is simply connected, that is, that the boundary of the face can be continuously shrunk to a single point without leaving the surface.

In § 10.3 we used a Schlegel diagram to prove Euler's formula. We could just as well have used the corresponding map on a sphere. The same argument, applied to a map on the general surface, shows that the Euler-Poincaré characteristic

$$\chi = V - E + F$$

is essentially a property of the surface, that is, that it has the same value for all maps drawn on the given surface. It is a remarkable fact that this property of the surface can be expressed very simply in terms of the integral curvature (which is obtained by integrating K over the whole surface).

Consider first a sphere, on which a map (with $V = E = 3$ and $F = 2$) is obtained by taking, as vertices, three points on a great circle. Each hemisphere is bounded by three arcs of the great circle, forming a spherical "triangle" whose three angles are π, π, π. By 20.31, each hemisphere has integral curvature

$$\pi + \pi + \pi - \pi = 2\pi.$$

Hence the integral curvature of the whole sphere is 4π (as it obviously must be, since the "spherical image" of any sphere is a unit sphere, whose surface area is 4π). The general formula, of which this is a very special case, is

20.41 $$\iint K\,dS = 2\chi\pi,$$

where the integration is taken over the whole of any given surface of characteristic χ.

To establish 20.41, we consider a map formed by E geodesic arcs on the given surface, choosing the V vertices in such positions that no face has a re-entrant angle (i.e., an angle greater than π). The map can then be "triangulated" by selecting a new vertex inside each face and joining it by new geodesic arcs to all the vertices of that face. This procedure yields a new map having $V + F$ vertices and $2E$ triangular faces. Since the sum of all the angles of all these $2E$ triangles amounts to 2π for each of the $V + F$ vertices, the integral curvature of the whole surface is

$$\begin{aligned}\iint K\,dS &= \Sigma\,(A + B + C - \pi)\\ &= 2\pi(V + F) - 2E\pi = 2\pi(V + F - E)\\ &= 2\pi\chi.\end{aligned}$$

It follows that the integral curvature of a closed surface is not altered by topological transformation. For instance, the value 4π is maintained when a sphere is deformed into an ellipsoid or any other oval surface. The deformation may even be continued so as to bring in anticlastic regions.

EXERCISES

1. The torus 8.88 (where $a > b$) is constructed by revolving a circle of radius b about a line (in its plane) distant a from the center. On this surface we can draw two circles, of radii b and $a + b$, which are geodesics having just one common point. These form a map in which $V = F = 1$, $E = 2$. Hence the integral curvature is zero. (The positive integral curvature of the "outer" synclastic part of the torus is exactly balanced by the negative integral curvature of the "inner" anticlastic part.)

2. Describe two further geodesics on the torus so that the four geodesics make a map in which $V = F = 4$, $E = 8$.

20.5 SURFACES OF CONSTANT CURVATURE

> *When Gauss was nineteen his mother asked his mathematical friend Wolfgang Bolyai whether Gauss would ever amount to anything. When Bolyai exclaimed, "The greatest mathematician in Europe!" she burst into tears.*
>
> E. T. Bell [**1**, p. 252]

When we study a surface by means of the fundamental magnitudes of the first order and the consequent Christoffel symbols, we are treating it "intrinsically," exploring it like the hypothetical two-dimensional creature who could not imagine any direction outside the surface itself. Such a creature could measure distances by means of the formula 19.16, distinguish geodesics as the shortest paths from place to place, and measure the Gaussian curvature K at any point.

One of the most elegant approaches to non-Euclidean geometry is to regard the elliptic or hyperbolic plane as a nondevelopable surface which is homogeneous (all positions alike) and isotropic (all directions alike). Since the surface is homogeneous, its Gaussian curvature is constant. By using a suitable unit of distance, we may take the constant value of K to be 1 or -1 according as it is positive or negative. We shall find it convenient to use geodesic polar coordinates. Since the surface is homogeneous and isotropic, the expression 20.24 will be the same wherever we place the pole $u^1 = 0$ and the initial geodesic $u^2 = 0$, and g will be a function of u^1 alone, independent of u^2. The "straight lines" of the geometry are the geodesics on the surface, and *it is not necessary to regard the surface as being embedded in a 3-space.*

Setting $K = \pm 1$ in 20.32, we obtain the differential equation

$$(\sqrt{g})_{11} = \mp \sqrt{g},$$

which yields

$$\sqrt{g} = A \sin u^1 + B \cos u^1 \quad \text{or} \quad A \sinh u^1 + B \cosh u^1.$$

At the pole, $ds\,(= du^1)$ is independent of u^2; therefore $g = 0$ when $u^1 = 0$; that is, $B = 0$. Also, by 20.27, $A = 1$. Hence

$$\sqrt{g} = \sin u^1 \quad \text{or} \quad \sinh u^1$$

and

$$ds^2 = (du^1)^2 + \sin^2 u^1\,(du^2)^2 \quad \text{or} \quad (du^1)^2 + \sinh^2 u^1\,(du^2)^2.$$

EXERCISE

Compute the circumference of the geodesic circle $u^1 = r$ (i) in the elliptic plane, (ii) in the hyperbolic plane.

20.6 THE ANGLE OF PARALLELISM

In the elliptic case, we can identify u^1 and u^2 with colatitude and longitude on the unit sphere (as in Ex. 2 at the end of § 20.3). Accordingly, we now restrict consideration to the hyperbolic case, in which

$$g = \sinh^2 u^1.$$

For convenience, let us write r, θ for u^1, u^2, so that*

$$ds^2 = dr^2 + \sinh^2 r \, d\theta^2.$$

To determine the straight lines of the hyperbolic plane, we use the differential equation 20.26 with $g = \sinh^2 r$, namely,

$$\frac{d}{ds}(\sinh^2 r \, \dot{\theta}) = 0.$$

For a first integral we obtain $\sinh^2 r \, \dot{\theta} = h^{-1}$ (a constant), whence

$$dr^2 + \sinh^2 r \, d\theta^2 = ds^2 = (h \sinh^2 r \, d\theta)^2,$$

so that $\quad dr^2 = \sinh^2 r \, (h^2 \sinh^2 r - 1) d\theta^2 \quad$ and

$$\theta = \int \frac{dr}{\sinh r \sqrt{h^2 \sinh^2 r - 1}} = \int \frac{\operatorname{csch}^2 r \, dr}{\sqrt{h^2 - \operatorname{csch}^2 r}}$$

$$= -\int \frac{d \coth r}{\sqrt{h^2 + 1 - \coth^2 r}} = \theta_0 + \arccos (k \coth r),$$

where $k = 1/\sqrt{h^2 + 1}$. Hence, finally, the lines are given by

$$k \coth r = \cos(\theta - \theta_0).$$

When h tends to infinity, so that k tends to zero, we have the radial lines, for which θ is constant. The line through $(a, 0)$ perpendicular to the initial line $\theta = 0$ (being a geodesic which is unchanged when θ is replaced by $-\theta$), is

$$\tanh a \coth r = \cos \theta.$$

We can use these results to find relations between the sides and angles of a right-angled triangle ABC with its right angle at C and its side BC along the initial line, as in Figure 20.6a. Since the equations for the lines BC, AB, and CA are

$$\theta = 0, \quad \theta = B, \quad \text{and} \quad \tanh a = \tanh r \cos \theta,$$

the lengths of the sides $BC = a$ and $AB = c$ are related by the equation

$$\tanh a = \tanh c \cos B$$

[Carslaw **1**, p. 109; Coxeter **3**, pp. 238, 282]. (Another formula of the same kind can be obtained by changing a and B into b and A.)

* E. Beltrami, *Giornale di Matematiche*, **6** (1868), p. 298 (12).

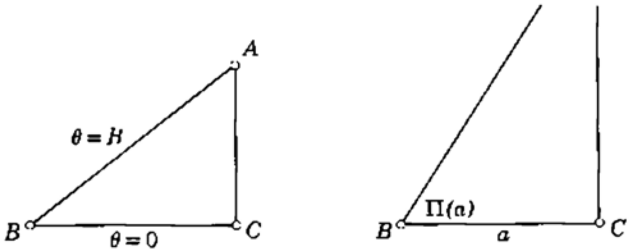


Figure 20.6a

The angle of parallelism $\Pi(a)$ is the value of B that makes c infinite; that is,

$$\Pi(a) = B,$$

where

$$\begin{aligned} \cos B &= \tanh a, \qquad \sin B = \operatorname{sech} a, \\ \csc B &= \cosh a, \qquad \cot B = \sinh a, \\ \csc B - \cot B &= \cosh a - \sinh a, \\ \tan \tfrac{1}{2}B &= e^{-a}. \end{aligned}$$

We have thus established Lobachevsky's formula

$$\Pi(a) = 2 \arctan e^{-a}$$

[Coxeter **3**, p. 208]. This is a precise expression for the function that we studied tentatively in § 16.3.

EXERCISE

Compute $\Pi(a)$ for a few suitably chosen values of a, and sketch the curve $y = \Pi(x)$. Where does $\Pi(u)$ occur in Figure 17.4b?

20.7 THE PSEUDOSPHERE

Having obtained the hyperbolic plane as a surface of constant negative curvature, it is natural for us to ask whether such a surface can be embedded in Euclidean space. In other words, can the hyperbolic plane, or a finite part of it, be represented isometrically by a surface in ordinary space, in some such manner as the elliptic plane is represented (twice over) by a sphere? The answer is No and Yes: there is no such representation of the whole hyperbolic plane* but there are certain surfaces that will serve for a portion of finite area [Klein **4**, p. 286]. The simplest instance, which Liouville named the *pseudosphere*, is one half of the "tractroid" formed by revolving the tractrix 17.51 about its asymptote.

Writing z for x, x for y, and setting $a = 1$, we obtain the tractrix

$$x = \operatorname{sech} u^1, \quad z = u^1 - \tanh u^1$$

* G. Lütkemeyer, *Ueber den analytischen Charakter der Integrale von partiellen Differentialgleichungen* (Göttingen, 1902).

in the plane $y = 0$. Revolution about the z-axis yields the tractroid

$$x = \operatorname{sech} u^1 \cos u^2, \qquad y = \operatorname{sech} u^1 \sin u^2, \qquad z = u^1 - \tanh u^1,$$

which has a cuspidal edge where $u^1 = 0$. The pseudosphere is the horn-shaped surface given by $u^1 \geqslant 0$. Differentiating the position vector $\mathbf{r} = (x, y, z)$, we obtain

$$g_{11} = \tanh^2 u^1, \qquad g_{12} = 0, \qquad g_{22} = \operatorname{sech}^2 u^1.$$

Setting $g_1 = \tanh u^1$, $g_2 = \operatorname{sech} u^1$ in 20.17, we deduce

$$-\tanh u^1 \operatorname{sech} u^1 K = (-\operatorname{sech} u^1)_1 = \operatorname{sech} u^1 \tanh u^1,$$

whence
$$K = -1.$$

Since the pseudosphere has the same Gaussian curvature as the hyperbolic plane, the geodesics on the former represent lines in the latter isometrically. Among these geodesics are the meridians (for which u^2 is constant), representing a pencil of parallels. Orthogonal to them, we find the circles for which $u^1 =$ constant, representing arcs of concentric horocycles (§ 16.6). The longest of these horocyclic arcs is of length 2π, since it is represented by the circle

$$x = \cos u^2, \qquad y = \sin u^2$$

in the plane $u^1 = 0$. Thus the whole pseudosphere represents the *horocyclic sector* bounded by this arc of length 2π and the diameters at its two ends. The horocyclic sector is wrapped round the pseudosphere so that the two diameters are brought together to form a single meridian.

This limitation to a horocyclic sector renders the pseudosphere utterly useless as a means for drawing significant hyperbolic figures. Every geodesic that is not merely a meridian winds itself round the "horn" as it proceeds in one direction, whereas in the opposite direction it is abruptly cut off by the cuspidal edge. Thus we cannot even draw such a simple arrangement of lines as Figure 16.3*a*! These remarks are needed to counteract the widespread but mistaken idea that hyperbolic geometry can be identified with the intrinsic geometry of the pseudosphere.

EXERCISE

Use 20.23 to obtain an equation for the geodesics on the pseudosphere.

21

Topology of surfaces

In Chapter 4 we considered various tessellations of the Euclidean plane (including, in § 4.6, *regular* tessellations). These may be regarded as infinite "maps." In § 15.7 we considered the analogous tessellations of a sphere, which are finite maps. In § 10.3 we proved Euler's formula

$$V - E + F = 2,$$

which connects the numbers of vertices, edges (or arcs), and faces (or regions) of any map drawn on a sphere. In § 20.4 we extended this to

$$V - E + F = \chi \leqslant 2$$

for a map on any closed surface, the Euler-Poincaré characteristic χ being the same for all maps on the given surface. In § 6.9 we identified antipodal points of a sphere so as to obtain the real projective plane; for a centrally symmetrical map on the sphere, this identification naturally halves V, E, and F, thus reducing χ from 2 to 1. In Figure 10.5*a* we considered reciprocal polyhedra which, when regarded as spherical tessellations, are a special case of *dual* maps. In § 10.1 we defined the Schläfli symbol $\{p, q\}$, which is appropriate for a map of p-gons, q at each vertex; and in 10.31 we obtained the equations $qV = 2E = pF$. In § 15.4 we discussed groups of permutations of the faces of a map. In § 15.3 we found that the theory of translations and glide reflections belongs to *absolute* geometry; that is, that it belongs not only to Euclidean geometry but also to hyperbolic geometry.

The present chapter applies all these ideas to a discussion of the topological properties of surfaces, including the conjecture of P. J. Heawood that, for the coloring of any map on a surface of characteristic χ,

$$[\tfrac{7}{2} + \tfrac{1}{2}\sqrt{49 - 24\chi}\,]$$

colors suffice. What makes this conjecture remarkable is that, although in 1890 he established its truth for every $\chi < 2$, it still remains an open question for maps on the ordinary sphere or plane. Another conjecture is that Heawood's formula is "best possible" in the sense that, for each χ, a map requiring the full number of colors can be drawn.

21.1 ORIENTABLE SURFACES

The group of transformations of greatest importance in present-day mathematics, namely, the group of topological transformations, is far wider than the projective group. Here we are dealing with a topological space; that is, with a set of elements, call them points, for which the concept of a neighborhood is defined. . . . Any transformation preserving neighborhoods is called topological.

Topology may be visualized as rubber-sheet geometry, since a topological transformation permits any amount of stretching or compressing (without tearing).

S. H. Gould [**1**, p. 304]

In Chapter 5 we mentioned Klein's famous classification of geometries according to the groups of transformations under which their theorems remain true. In this sense, projective geometry is characterized by the group of collineations and correlations, and hyperbolic geometry by the subgroup of collineations leaving invariant a conic (the locus of points at infinity). Topology, sometimes described as "the most general of all geometries," is characterized by the group of *continuous* transformations. For instance, since a polyhedron can be continuously transformed into the corresponding spherical tessellation, topology does not recognize any distinction between the polyhedron and the tessellation. Again, in § 20.4 we defined the characteristic

21.11 $$\chi = V - E + F$$

in terms of a map formed by geodesic arcs on the given surface, but the value of χ will not change if we replace the geodesic arcs by any continuous arcs which join the same pairs of points without crossing one another. In other words, the edges of the map are not necessarily geodesics like the boundary between Colorado and Utah; they can just as well be "wild" like the boundary between Indiana and Kentucky.

In the same spirit, the torus 8.88 is topologically equivalent to a sphere with a *handle* (like the handle of a teacup), and we can derive more compli-

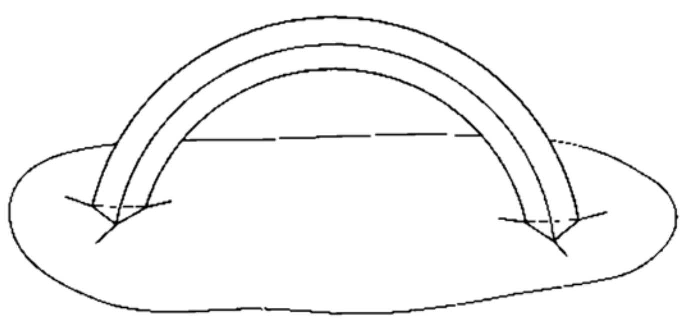


Figure 21.1a

cated surfaces by adding any number of further handles. The operation of adding a handle to a given surface reduces the value of χ by 2. For, since the handle may be chosen in the form of a bent triangular prism joining two triangular faces of a suitable map, as in Figure 21.1*a*, its insertion leaves V unchanged, increases E by 3, and increases F by $3 - 2$. Knowing that a a sphere has $\chi = 2$, we deduce that a sphere with p handles has

21.12 $$\chi = 2 - 2\text{p}.$$

This is called a surface of *genus* p. In particular, a sphere is a surface of genus 0 and a torus is a surface of genus 1.

From a rectangular rubber sheet, we can make a model of a torus by identifying, or bringing together, each pair of opposite sides. The first identification produces a tube, and the second an "inner tube." Conversely, by cutting a torus along two circles that have only one common point, we can unfold it (after some distortion) to make a rectangle whose pairs of opposite sides arise from the two cuts. More generally, given a surface of genus p, a suitable set of 2p cuts, all beginning and ending at a single point, enables us to unfold the surface into a 4p-gon whose pairs of opposite sides arise from the 2p cuts [Coxeter and Moser **1**, p. 25], as in Figure 21.1*b*.

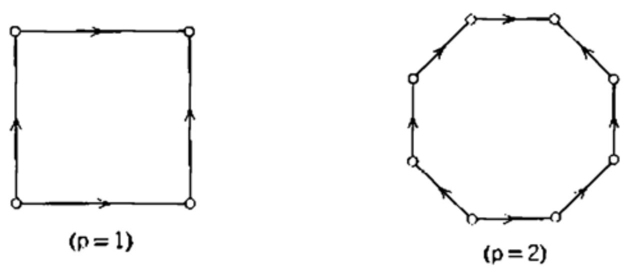


Figure 21.1b

If we regard the 2p cuts on the surface as the 2p edges of a map, we find that this map has one face and one vertex, in agreement with the formula

$$\chi = V - E + F = 1 - 2\text{p} + 1 = 2 - 2\text{p}.$$

When p $= 1$, the rectangle is conveniently taken to be a square, and this square may be regarded as one of the infinitely many faces of the regular tessellation $\{4, 4\}$ (Figure 4.6*a*) or as the unit cell of a lattice generated by two translations in perpendicular directions. The identification of opposite sides may be achieved by identifying points in corresponding positions in all the squares, that is, by pretending that the translations have no effect. In technical language, the Euclidean plane is the *universal covering surface* of the torus. Similarly, when p > 1, the 4p-gon is conveniently taken to be a regular 4p-gon of angle $\pi/2$p in the hyperbolic plane, and this $\{4\text{p}\}$ may be regarded as a face of a regular hyperbolic tessellation $\{4\text{p}, 4\text{p}\}$. Opposite sides of the $\{4\text{p}\}$ are related by 2p translations, which generate a group hav-

ing the $\{4p\}$ for a fundamental region. The infinite hyperbolic plane (which is the universal covering surface) is reduced to the given finite surface by identifying each point in one $\{4p\}$ with the corresponding points in all the other $\{4p\}$'s. The group of translations is called the *fundamental group* of the surface [Coxeter and Moser **1**, pp. 24–27, 58–60].

EXERCISE

A *graph* is a set of points (called *vertices*), certain pairs of which are joined by arcs (called *edges*). In particular, the vertices and edges of a map form a graph. Conversely, any connected graph can be drawn on a surface so as to form a map covering the surface.* A graph is said to be *planar* if it can be drawn on a sphere without any edges crossing one another, in which case it can just as easily be drawn in the (inversive) plane, provided we allow one face of the consequent map to be infinite. A vertex is said to have *valency* (or "degree") q if it belongs to q edges. A graph is said to be *trivalent* if every vertex belongs to three edges. In this case $3V = 2E$; therefore V is even. The *Thomsen graph*† has six vertices $P_1, \ldots, P_6$ and nine edges P_iP_j, where $i + j$ is odd. This is the simpest nonplanar trivalent graph. Can it be drawn on a torus?

21.2 NONORIENTABLE SURFACES

A surface is non-orientable if and only if there exists on the surface some closed curve . . . such that a small oriented circle whose center traverses the curve continuously will arrive at its starting point with its orientation reversed.

Hilbert and Cohn-Vossen
[**1**, p. 306]

Each of the surfaces discussed in § 21.1 is *orientable*, that is, a positive sense of rotation can be defined consistently everywhere. More precisely, the faces of any map on the surface can be regarded as directed polygons in such a way that the two directions thus assigned to each edge disagree, or cancel out. A surface is said to be *nonorientable* if it admits one map which cannot be oriented in this manner. The most famous instance is the *Möbius strip*, which can be illustrated by taking a strip of paper $ABAB$, several times longer than it is wide, and sticking the two ends together after twisting one of them by a half-turn. Its nonorientability can be checked by means of a map consisting of a single row of squares. It is *one-sided* in the sense that an ant could crawl along the whole length of the strip, without crossing the bounding edge, and find himself at the starting point on the "other side." If two wheels in a machine are connected by a belt of such a shape (e.g., for the purpose of conveying hot or abrasive materials), the substance of the belt will wear out equally on both sides. A patent for

* J. H. Lindsay, Jr., Elementary treatment of the imbedding of a graph in a surface, *American Mathematical Monthly*, **66** (1959), pp. 117–118.

† W. Blaschke and G. Bol, *Geometrie der Gewebe* (Berlin, 1938), p. 35.

this practical application of the Möbius strip has been acquired by the Goodrich Company.*

Unlike the closed surfaces considered in § 21.1, the Möbius strip is *bounded.* The boundary is a simple closed curve, but physically it cannot be shrunk away because, if we make it coincide with the circumference of a circle, the interior of the circle must intersect the surface of the strip. This practical difficulty arises because the model is embedded in Euclidean space. Theoretically, no such embedding is needed. When the boundary has been shrunk away, the resulting closed surface is topologically a real projective plane! In other words, *the Möbius strip is the real projective plane with a hole cut out of it.* For we may regard the projective plane (§ 6.9) as a sphere with antipodal points identified. When cutting out a circular hole round the north pole we must, of course, also cut out an equal hole round the south pole. What remains of the sphere is a zone bounded by two parallels of latitude such as the Tropics of Cancer and Capricorn. But the identification of antipodes has the effect that only half the zone is needed, say the "visible" half (Figure 21.2*a*). This half-zone, with its ends AB identified, is evidently a Möbius strip.

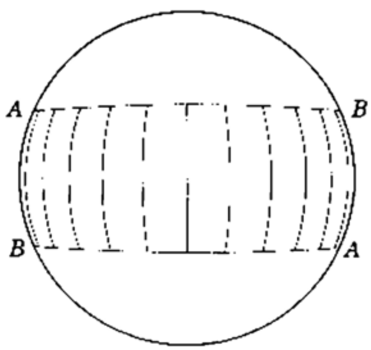


Figure 21.2a

Instead of a whole sphere with every pair of antipodal points identified, we may regard the projective plane as a hemisphere (say the "southern" hemisphere) with identification of diametrically opposite points on the peripheral equator. In the spirit of topology, the hemispherical surface can be stretched until it covers almost the whole sphere, and the periphery (with opposite points identified) is reduced to a very small circle round the north pole. In other words, the projective plane is topologically equivalent to a sphere with a *cross-cap,* which may be described as a small circular hole having the magic property that, as soon as the crawling ant reaches it, he finds himself leaving the same hole from its diametrically opposite point (inside, instead of outside, the sphere).

* U.S. Patents 1,442,632 (1923), 2,479,929 (1949), 2,784,834 (1957).

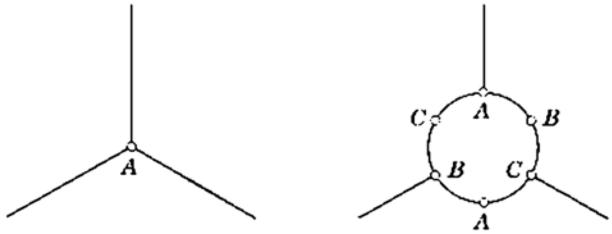


Figure 21.2b

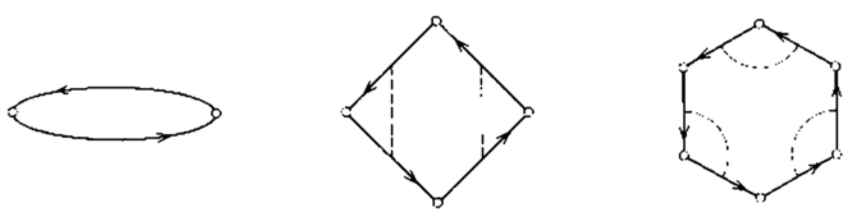
We can derive more complicated surfaces by adding any number of cross-caps, each of which reduces the value of χ by 1. For, if a map on a given surface has a vertex A belonging to three faces, we can replace A by a cross-cap $ABCABC$ as in Figure 21.2*b*. Since this requires the formation of two new vertices B, C, and three new edges BC, CA, AB, the insertion of the cross-cap increases V by 2, E by 3, and leaves F unchanged. (The faces on the left and right now meet twice: along the original edge through A and again along the new edge BC.) Knowing that a sphere has $\chi = 2$, we deduce that a sphere with q cross-caps has

21.21 $$\chi = 2 - \mathrm{q}.$$

When q = 1 this is, as we have seen, the real projective plane. When q = 2 it is the *Klein bottle* (or "nonorientable torus") [Hilbert and Cohn-Vossen **1**, p. 308].

A suitable set of q cuts, all beginning and ending at a single point and each passing through a different cross-cap, enables us to unfold the surface into a 2q-gon such that q pairs of adjacent sides arise from the q cuts [Coxeter and Moser **1**, pp. 25–28, 56–58] as in Figure 21.2*c*. These cuts are the q edges of a map having one face and one vertex, in agreement with the formula

$$\chi = V - E + F = 1 - \mathrm{q} + 1 = 2 - \mathrm{q}.$$



(q = 1) (q = 2) (q = 3)

Figure 21.2c

When q = 1, the 2q-gon is a digon which may be regarded as one of the two faces of the spherical tessellation {2, 2} (see Ex. 1 at the end of § 15.7). In fact, the universal covering surface of the projective plane is the sphere, and its fundamental group is of order 2, generated by the central inversion.

When q = 2, the 2q-gon may be regarded as a face of the Euclidean tes-

sellation $\{4, 4\}$, so that the universal covering surface is the Euclidean plane. Unlike the torus, whose fundamental group **p1** is generated by two translations, the Klein bottle has the fundamental group **pg**, generated by two glide reflections (see § 4.3 and Plate I). Similarly, when $q > 2$, so that the universal covering surface is the hyperbolic plane, the 2q-gon is a face of the hyperbolic tessellation $\{2q, 2q\}$, and the fundamental group is generated by q glide reflections [Coxeter and Moser **1**, pp. 56–58].

EXERCISES

1. The projective plane is topologically equivalent to a disk* with diametrically opposite points identified.

2. How can the Thomsen graph (see the end of § 21.1) be drawn on a sphere with a cross-cap (or a disk with opposite points identified)?

3. What happens to the vertices and edges of a regular hexagonal prism when we project it centrally onto its circumsphere and then identify antipodes?

4. Is a sphere with p handles and q cross-caps topologically equivalent to a sphere with $2p + q$ cross-caps?

21.3 REGULAR MAPS

We first give a method of reducing any two-dimensional manifold to one of the known polygonal normal forms. The method used is one by which a polygon on which the manifold is represented is subjected to a series of transformations by cutting it apart in a simple manner and then joining it together again so as to obtain a new polygon representing the same manifold.

H. R. Brahana (1895-)

(*Annals of Mathematics*, **23** (1921), p. 144)

It can be proved† that every closed surface is topologically equivalent either to a sphere with p ($\geqslant 0$) handles (if the surface is orientable) or to a sphere with q (> 0) cross-caps. In virtue of 21.12 and 21.21, this means that, from the standpoint of topology, there is just one orientable closed surface for each of the values

$$\chi = 2, \quad 0, -2, -4, \ldots,$$

namely a sphere with $1 - \frac{1}{2}\chi$ handles, and there is just one nonorientable closed surface for each of the values

$$\chi = 1, \quad 0, -1, -2, \ldots,$$

* A disk is a circle plus its interior. For other topological properties of the disk and sphere, see A. W. Tucker, *Proceedings of the First Canadian Mathematical Congress* (University of Toronto Press, 1946).

† Brahana's original proof has been simplified by Lefschetz [**1**, pp. 72–85] and others. One of the best expositions is by R. C. James, Combinatorial topology of surfaces, *Mathematics Magazine*, **29** (1955), pp. 1-39.

namely, a sphere with $2-\chi$ cross-caps. We have already described, on such a surface, a very simple map having one vertex, $2-\chi$ edges, and one face. In the orientable case, this map is "regular" in the following sense.

The vertices, edges, and faces of a map (on any closed surface, orientable or nonorientable) may conveniently be called the *elements* of the map. Those permutations of the elements which preserve all the relations of incidence are called *automorphisms* of the map. The automorphisms form a group (of order 1 or more) called *the group of the map*. This is a natural generalization of the symmetry group of a polyhedron or tessellation (§ 15.7), but metrical ideas are no longer used. A map is said to be *regular* if its automorphisms include the cyclic permutation of the edges (and vertices) belonging to any one face and also the cyclic permutation of the edges (and faces) that meet at any one vertex of this face. Such a map is "of type $\{p, q\}$" if p edges belong to a face, and q to a vertex. The *dual* map, whose edges cross those of the original map, is of type $\{q, p\}$. (The letters p and q used here have no connection with our previous use of p and q for the numbers of handles and cross-caps.)

The equations 10.31 remain valid. Combining them with 21.11, we obtain a generalization of 10.32:

21.31 $$V = 2pr, \quad E = pqr, \quad F = 2qr,$$

where, if $\chi \neq 0$,

21.32 $$r = \frac{\chi}{2p + 2q - pq}.$$

If $\chi = 0$, so that $2p + 2q = pq$ as in § 4.6, there are infinitely many possible values for r, as we shall soon see.

If $\chi = 1$ or 2, the possible values for p and q are given by 10.33 without the restrictions $p > 2$, $q > 2$. Thus the regular maps on a sphere ($\chi = 2$) are just the spherical tessellations

21.33 $$\begin{array}{llll} \{p, 2\}, & \{2, p\}, & \{3, 3\}, & \\ \{4, 3\}, & \{3, 4\}, & \{5, 3\}, & \{3, 5\}, \end{array}$$

namely: the *dihedron* whose p vertices are evenly spaced along the equator, the *hosohedron** whose edges and faces are p meridians and p lunes, and "blown-up" variants of the five Platonic solids. All these are centrally symmetrical, except the dihedron and its dual with p odd, and the tetrahedron $\{3, 3\}$. In the centrally symmetrical cases we can identify antipodes to obtain the regular tessellations of the elliptic plane ($\chi = 1$):

21.34 $$\begin{array}{llll} \{p, 2\}/2 & \text{and} & \{2, p\}/2 & (p \text{ even}), \\ \{4, 3\}/2, & \{3, 4\}/2, & \{5, 3\}/2, & \{3, 5\}/2 \end{array}$$

[Coxeter and Moser **1**, p. 111]. For instance, identifying opposite elements

* This term (literally "any number of faces") was coined by Vito Caravelli (1724–1800), whose *Traité des hosoèdres* was published in Paris (1959) by the Librairie Scientifique et Technique.

of the cube (Figure 10.5*b*), we see that $\{4, 3\}/2$ is a partition of the elliptic plane (or of the real projective plane) into three "squares," say $ABCD$, $ACDB$, $ADBC$. (These squares are just the three handkerchiefs which Lady Muriel began to sew together in her attempt to make the Purse of Fortunatus [Dodgson **4**, pp. 100–104]. The first two, joined along their common side CD, form a Möbius strip whose boundary is $ADBC$.) Likewise, $\{5, 3\}/2$ is a partition of the elliptic plane into six pentagons, each of which is surrounded by the remaining five.

The regular maps on the torus are derived from infinite regular maps on its universal covering surface, which is the Euclidean plane. As we saw in § 4.6, these infinite maps are the regular tessellations

21.35 $$\{6, 3\}, \quad \{4, 4\}, \quad \{3, 6\}.$$

The necessary identifications are determined by subgroups of the translation groups of these tessellations.

The vertices of $\{4, 4\}$ may be taken to be the lattice of points whose Cartesian coordinates (x, y) are integers. The torus is derived by identifying opposite sides of a square, one of whose sides goes from $(0, 0)$ to (b, c), where b and c are positive integers or zero (but not *both* zero). Since the area of this square is $b^2 + c^2$, the part of the original $\{4, 4\}$ that lies inside it consists of $b^2 + c^2$ unit squares. We thus find, on the torus, a map

$$\{4, 4\}_{b,c}$$

in which

$$V = b^2 + c^2, \qquad E = 2V, \qquad F = V$$

[Coxeter and Moser **1**, p. 103]. In particular, $\{4, 4\}_{1,0}$ is the map (having one vertex and one face) which we used in § 21.1 when we unfolded the torus after cutting it along the two edges of the map. (If it seems paradoxical for a map of type $\{4, 4\}$ to have only one vertex, we must recognize that the face is still quadrangular even though its four vertices all coincide with the single vertex of the map.) The map $\{4, 4\}_{2,1}$, whose five faces are each surrounded by the remaining four, is shown in Figure 21.3*a*.

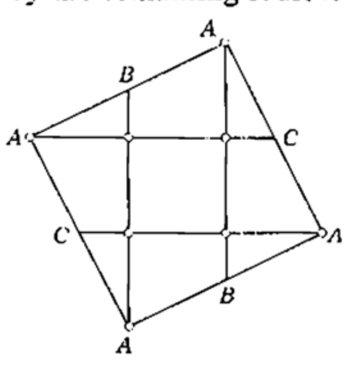


Figure 21.3a

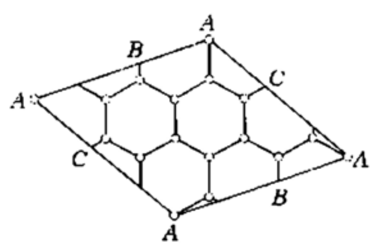


Figure 21.3b

Similarly, the vertices of $\{3, 6\}$ may be taken to be the lattice of points whose oblique coordinates, with axes inclined at 60°, are integers. The torus is derived by identifying opposite sides of a rhombus of angle 60°, one of whose sides goes from $(0, 0)$ to (b, c). Since the area of this rhombus is $b^2 + bc + c^2$ times that of the unit cell of the lattice (consisting of two adjacent faces of $\{3, 6\}$), the part of $\{3, 6\}$ that lies inside it consists of $2(b^2 + bc + c^2)$ equilateral triangles. We thus find, on the torus, a map

$$\{3, 6\}_{b,c}$$

in which

$$V = b^2 + bc + c^2, \qquad E = 3V, \qquad F = 2V$$

[Coxeter and Moser **1**, p. 107]. (For an affine variant of one-half of $\{3, 6\}_{2,1}$, see Figure 13.5*c*.) The dual map

$$\{6, 3\}_{b,c}$$

has $b^2 + bc + c^2$ hexagonal faces. In particular, $\{6, 3\}_{2,1}$ (Figure 21.3*b*)* is Heawood's partition of the torus into seven hexagons, each of which is surrounded by the remaining six.

Thus we see that the torus admits infinitely many regular maps of each of the three types 21.35. On the other hand, there are no regular maps on the Klein bottle [Coxeter and Moser **1**, p. 116].

If a regular map has more than one vertex and more than one face, every edge joins two vertices and separates two faces. If there is an automorphism which interchanges these two vertices without interchanging the two faces (in which case there is another automorphism which does vice versa), the map is said to be *reflexible* [Ball **1**, p. 129; Coxeter and Moser **1**, p. 101]. Clearly, all the regular maps on the sphere and all those on any nonorientable surface are reflexible, but those on the torus are reflexible only if $bc(b - c) = 0$. It was suggested by Coxeter and Moser [**1**, p. 102] that possibly all regular maps on more complicated surfaces (i.e., on surfaces of negative characteristic) are reflexible. However, this conjecture is refuted by J. R. Edmonds' discovery of a nonreflexible regular map† of type $\{7, 7\}$ and genus 7, having 8 vertices, 28 edges, and 8 heptagonal faces

$$FDCGBEH,\quad GEDACFH,\quad AFEBDGH,\quad BGFCEAH,\quad CAGDFBH,$$
$$DBAEGCH,\quad ECBFADH,\quad ABCDEFG.$$

EXERCISES

1. Describe the maps $\{2, 1\}$ and $\{1, 2\}$ on the sphere. (The former has one face, a digon $\{2\}$; the latter has two faces which are monogons $\{1\}$.)

2. The elliptic tessellation $\{2q, 2\}/2$ has q vertices and q edges, all on one line,

* For other ways of drawing Heawood's map, see Coxeter, Map-coloring problems, *Scripta Mathematica*, **23** (1957), pp. 19–21, and The four-color map problem, 1840–1890, *Mathematics Teacher*, **52** (1959), pp. 288–289.

† See also Robert Frucht, *Canadian Journal of Mathematics*, **4** (1952), p. 247.

and one face. Its dual, $\{2, 2q\}/2$, has one vertex, q complete lines for its edges, and q angular regions for its faces. Describe the remaining regular tessellations of the elliptic plane.

3. As we have seen, the standard decomposition of the torus provides a one-faced map $\{4, 4\}_{1,0}$. The standard decomposition of the Klein bottle (as in the second part of Figure 21.2*c*) provides another one-faced map of type $\{4, 4\}$, but this is not regular. In both cases, the one vertex and the two edges form a very simple graph, which may be described roughly as a figure of eight. The same graph can be drawn on the projective plane to form $\{2, 4\}/2$, or on the sphere to form an irregular map whose faces consist of a digon and two monogons.

4. The standard decomposition of a sphere with three cross-caps (as in the third part of Figure 21.2*c*) provides an irregular one-faced map of type $\{6, 6\}$. Its one vertex and three edges form a "clover-leaf" having three loops. The same graph can be drawn on the Klein bottle as an irregular map of type $\{3, 6\}$, on the torus as $\{3, 6\}_{1,0}$, on the projective plane as $\{2, 6\}/2$, and on the sphere as an irregular map whose faces consist of a triangle and three monogons.

5. Describe the reflexible maps

$$\{4, 4\}_{1,1}, \quad \{4, 4\}_{2,0}, \quad \{3, 6\}_{1,1}, \quad \{6, 3\}_{1,1}, \quad \{6, 3\}_{2,0}.$$

6. The vertices and edges of $\{6, 3\}_{1,1}$ form the Thomsen graph (see the end of § 21.1). Those of $\{4, 4\}_{2,2}$ form an analogous graph having eight vertices instead of six.

7. A graph is called a *complete V-point* if every two of its V vertices are joined by an edge. The vertices and edges of the following maps form complete V-points (for which values of V?):

$$\{3, 2\}, \quad \{3, 3\}, \quad \{4, 3\}/2, \quad \{4, 4\}_{2,1}, \quad \{3, 5\}/2, \quad \{3, 6\}_{2,1}.$$

8. The vertices and edges of Edmonds' map form a complete 8-point.

9. There is no map of type $\{1, 1\}$. (*Hint:* Set $p = q = 1$ in 21.31 and 21.32.)

21.4 THE FOUR-COLOR PROBLEM

"I doubt it," said the Carpenter,
And shed a bitter tear.

Lewis Carroll

[Dodgson **2**, Chap. 4]

The theory of maps on surfaces may be said to have begun in 1840, when Möbius puzzled his students with the problem of dividing a country into five districts in such a way that every two would have a common boundary line (not merely a common point). The impossibility of such a partition led naturally to the question whether four colors always suffice for coloring a map when we stipulate that different colors are needed wherever two districts share a boundary line or, in mathematical terms, wherever two faces share an edge. It must be emphasized that each face is simply connected (i.e.,

topologically a disk). Thus the geographical problem applies to a single island or continent: the ocean and all the other islands and continents are to be taken together as forming one more face, and if this is a blue face the same color must be allowed for some of the other faces. For instance, in a map of Europe, we need three different colors (say green, red, and yellow) for Belgium, France, and Germany; Holland may have the same color as France, but Luxembourg must be blue, like the sea.

Figures 15.4*a*, *b*, *c* illustrate the use of four colors for the tetrahedron and octahedron, and five for the icosahedron. For the tetrahedron, four is the only possible number, since each face meets all the others. Apart from this simplest case, *any map whose faces are triangles can be colored in three colors.** Moreover, any map having an even number of faces at each vertex (such as the octahedron) can be colored in two colors, like a chessboard.

The problem of deciding whether four colors suffice for coloring any map on a plane or a sphere is sometimes called *Guthrie's problem,* after Francis Guthrie, who took his B.A. in London in 1850 and his LL.D. in 1852. Between these dates the problem occurred to him while he was coloring a map of England. He tried in vain to prove that four colors are always sufficient. On October 23, 1852, his younger brother Frederick communicated the conjecture to Augustus De Morgan (author of *A Budget of Paradoxes*). In 1878, Cayley revived interest in the problem at a meeting of the London Mathematical Society by asking whether anyone had proved the conjecture. In 1880, Cayley's challenge was answered by A. B. Kempe and P. G. Tait, who published plausible arguments, which were accepted for ten years (even by Klein himself), as proving that four colors will always suffice. In 1890, Heawood drew attention to the fallacy in Kempe's argument, using for a counterexample a particular map having 18 faces. The number of faces can actually be reduced to 9, so as to reveal the fallacy more quickly.†

In §§ 21.5-21.7 we shall describe the valid part of Kempe's work and also Heawood's extension to maps on the torus and other multiply connected surfaces.

EXERCISES

1. In how many essentially different ways can a cube be colored with three given colors, a dodecahedron with four?

2. In Figure 15.4c, the icosahedron is colored with five colors so that each face and its three neighbors have four different colors. Replace each "*e*" by the one remaining color, thus reducing the number of colors to four. Starting afresh, color the icosahedron with three colors [Ball **1**, pp. 238–241].

3. Try to draw a map that is difficult to color with four colors.

* R. L. Brooks, On coloring the nodes of a network, *Proceedings of the Cambridge Philosophical Society*, **37** (1941), pp. 194–197.

21.5 THE SIX-COLOR THEOREM

Even the moonlit track ahead of him faded from his consciousness, for into his head had come a theorem which might be true or might be false, and his mind darted hither and thither seeking proofs to establish its truth and counter-examples to show that it could not possibly be true.

J. L. Synge [**2**, p. 165]

The equations 10.31, which apply to a regular map of type $\{p, q\}$, remain valid for a general map having various kinds of face and various numbers of faces at a vertex, provided we interpret p as the *average* number of vertices (or edges) of a face, and q as the *average* number of faces (or edges) at a vertex. Since a vertex belonging to only two edges can be omitted by combining the two edges, there is no real loss of generality in assuming that every vertex belongs to at least three edges. Thus $q \geqslant 3$ and

$$2E = qV \geqslant 3V,$$

whence, by 21.11, $E \leqslant 3(E - V) = 3(F - \chi)$ and

21.51 $$p = \frac{2E}{F} \leqslant 6\left(1 - \frac{\chi}{F}\right).$$

This proves that $p < 6$ whenever $\chi > 0$, that is, for the sphere ($\chi = 2$) or the projective plane ($\chi = 1$). Hence

21.52 *Every map on the sphere or the projective plane has at least one face whose number of edges is less than* 6.

We can now prove, by induction over the number of faces,

THE SIX-COLOR THEOREM. *To color any map on the sphere or the projective plane requires at most six colors.*

We take "any map" to mean "any map having F faces," for each particular value of F. When $F \leqslant 6$ there is no problem: we can assign distinct

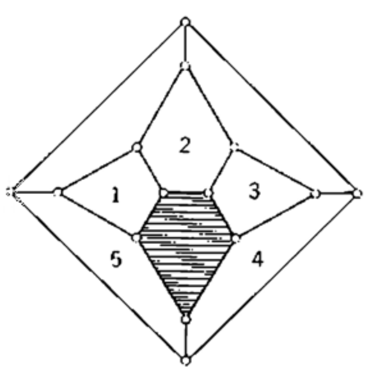


Figure 21.5a

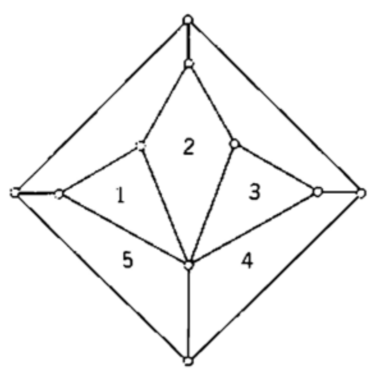


Figure 21.5b

colors to all the faces. The theorem will be proved if we can deduce the case of 7 faces from the case of 6, and then 8 from 7, and so on. Accordingly, we make the inductive assumption that the theorem holds for every map of $F - 1$ faces, and then proceed to investigate a given F-faced map, paying particular attention to the face (or one of the faces) having 5 or fewer edges (see 21.52). For definiteness, we assume this face to be a pentagon, like the shaded face in Figure 21.5*a*. (The same arguments can be carried through with trivial changes if it is a quadrangle, triangle, or digon.) Figure 21.5*b* shows a modified map in which this pentagonal face has shrunk to a point, that is, in which its territory has been ceded to its five neighbors. By the inductive assumption, the modified map, having only $F - 1$ faces, can be colored with six colors. Let this be done, and let the same coloring be applied to the original map. Then, even if the five neighbors need five distinct colors, there is still a sixth color left for the pentagonal face itself.

Since this argument can be applied with $F = 7$, then with $F = 8$, and so on, the six-color theorem holds for all values of F.

Can the number 6 be replaced by 5? For the projective plane it cannot, as we shall soon see. For the sphere it can, by a subtler argument depending on the topological theorem that a circle on the sphere decomposes it into two separate regions [Ball **1**, p. 229]. But the gap between the five colors that are always sufficient and the four that are usually necessary has never been bridged. Heawood himself continued to investigate the problem for the rest of his life, reducing it to pure algebra. Other authors have gradually increased the lower bound of the number of faces for a map that might possibly require five colors.

It is almost certainly a mere coincidence that the numbers 4 and 5 play an analogous role in arithmetic. According to Mordell [**1**, p. 19], it is "very easy to prove that every integer is the sum of at most five integer cubes, positive or negative, and there is an unproved conjecture that four cubes suffice."

For the projective plane, on the other hand, there is no gap to be bridged: six colors are both necessary and sufficient. The simplest map that needs

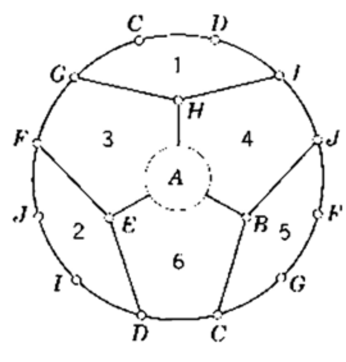


Figure 21.5c

all six is $\{5, 3\}/2$ (see p. 387), which is drawn in Figure 21.5c as a disk with diametrically opposite points identified. By cutting out a hole round the vertex A, we obtain H. Tietze's six-color map on the Möbius strip (Figure 21.5d). Here, as on the whole projective plane, six colors are both necessary and sufficient.

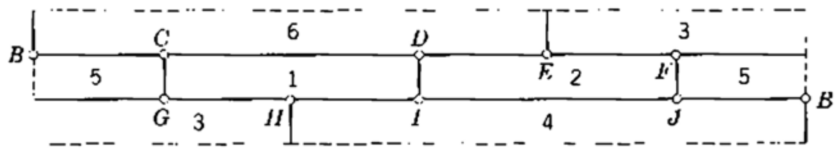


Figure 21.5d

EXERCISE

Make a model of the Möbius strip and color it as indicated in Figure 21.5d. (Since this is a "one-sided" surface, the paper must everywhere have the same color on both sides.)

21.6 A SUFFICIENT NUMBER OF COLORS FOR ANY SURFACE

> [Huck Finn to Tom Sawyer in their flying boat:] *"We're right over Illinois yet. And you can see for yourself that Indiana ain't in sight. . . . Illinois is green, Indiana is pink. You show me any pink down here, if you can. No, sir; it's green."*
>
> *"Indiano pink? Why, what a lie!"*
>
> *"It ain't no lie; I've seen it on the map, and it's pink."*
>
> Mark Twain (=S. L. Clemens, 1835-1910)
>
> (*Tom Sawyer Abroad*, Harper, New York, 1896, Chap. 3)

The problem of coloring maps on a more complicated surface is not difficult, as on the sphere, but easy, as on the projective plane. In fact, we can now prove

HEAWOOD'S THEOREM. *To color any map on a surface of characteristic $\chi < 2$ requires at most $[N]$ colors, where*

$$N = \frac{7 + \sqrt{49 - 24\chi}}{2}.$$

Since the case $\chi = 1$ has already been proved in § 21.5, we shall suppose that

$$\chi \leqslant 0.$$

Since the theorem is obviously true when $F \leqslant N$ (which implies $F \leqslant [N]$), we shall suppose also that

$$F > N$$

and use induction over the number of faces, assuming that $[N]$ colors suffice for any map having $F - 1$ faces. Since N satisfies the quadratic equation

$$N^2 - 7N + 6\chi = 0$$

or

$$6\left(1 - \frac{\chi}{N}\right) = N - 1,$$

the inequality 21.51 yields

$$p \leqslant 6\left(1 - \frac{\chi}{F}\right) \leqslant 6\left(1 - \frac{\chi}{N}\right) = N - 1.$$

Hence there is at least one face having $[N] - 1$ or fewer edges (cf. 21.52). We continue as in § 21.5, using $[N]$ instead of 6, and conclude that $[N]$ colors suffice for the given map.

Although this proof would break down if χ were positive, we note that Heawood's expression for N not only yields the correct value 6 when $\chi = 1$ but also yields the conjectured value 4 when $\chi = 2$.

EXERCISE

Tabulate $[N]$ for values of χ from 2 down to -9.

21.7 SURFACES THAT NEED THE FULL NUMBER OF COLORS

"Suppose there's a brown calf and a big brown dog, and an artist is making a picture of them. . . . He has got to paint them so you can tell them apart the minute you look at them, hain't he? Of course. Well, then, do you want him to go and paint both of them brown? Certainly you don't. He paints one of them blue, and then you can't make no mistake. It's just the same with maps. That's why they make every state a different color. . . ."

Mark Twain (*ibid.*)

Heawood's theorem (with $\chi = 0$) tells us that every map on the torus can be colored with seven colors. His regular map $\{6, 3\}_{2,1}$, whose seven faces all meet one another, shows that at least one map on the torus really needs seven colors. Since Heawood's expression for N depends only on χ, it yields the same number 7 for the Klein bottle. However, Philip Franklin has proved a *six-color theorem for the Klein bottle.**

Ringel [**1**, p. 124] proved that the Klein bottle is the *only* nonorientable surface not needing as many as $[N]$ colors. For instance, inserting a cross-cap at one vertex of Heawood's seven-faced map on the torus, as in Figure 21.2*b*, we obtain a seven-faced map on the surface of characteristic -1. This map still needs seven colors, since all its faces meet one another. (In fact, some pairs of faces meet twice.)

* Coxeter, *Scripta Mathematica*, **23** (1957), pp. 21–23.

In the case of an orientable surface of genus $p = 1 - \frac{1}{2}\chi$, Heawood's number $[N]$ is given by

$$N = \frac{7 + \sqrt{48p + 1}}{2},$$

and it is appropriate to give the name "The Heawood conjecture" to the statement that, for every p, the orientable surface of genus p carries a map of $[N]$ faces, all meeting one another. This conjecture became a theorem in 1968, when Ringel and Youngs* found such a map for every p. This breakthrough was the climax of a long story. The investigation was begun by L. Heffter in 1891 and was then neglected until 1952, when Ringel resumed it. He collaborated with Youngs from 1966 on, and some helpful ideas were contributed by W. Gustin, C. M. Terry, and L. R. Welch. The values of p up to 32 were disposed of independently by Jean Mayer (a professor of French literature) in 1967. The four most difficult cases ($p = 59, 83, 158, 257$) were achieved by Youngs and Richard Guy in 1968.

EXERCISES

1. Replacing the "hole" in Figure 21.5*c* by a cross-cap, obtain a six-color map (of 3 pentagons and 3 heptagons) on the Klein bottle.

2. Draw an eight-color map on the surface of genus two [Ball **1**, p. 237].

* Gerhard Ringel and J. W. T. Youngs, Solution of the Heawood map-colouring problem, *Proceedings of the National Academy of Sciences* (U.S.A.), 1968.

22

Four-dimensional geometry

The idea of four-dimensional space has long been surrounded by an attractive aura of mystery. The axiomatic approach (12.44) dispels the mystery without reducing the fascination. Having become accustomed to non-Euclidean geometries, we are no longer disconcerted by the possibility that two planes may have a common point without having a common line. More simply, we may regard the points of Euclidean 4-space as having four Cartesian coordinates instead of the usual two or three. Any two distinct points determine a line, the three vertices of a triangle determine a plane, and the four vertices of a tetrahedron determine a *hyperplane*, which is given by a single linear equation connecting the four coordinates.

In §§ 22.1–22.3 we describe the four-dimensional analogues of the Platonic solids. We shall see that there are six of these regular *polytopes*. Each consists of a finite number of solid cells in distinct hyperplanes, so arranged that every face of each cell belongs also to another cell. All these regular polytopes were discovered by Schläfli before 1855.

Just as we can make flat pictures of solids by projecting them orthogonally onto a plane, so we can make flat or solid "pictures" of hypersolids by projecting them either onto a plane or onto a hyperplane. Instances of the former procedure are shown in Figures 22.1*a*, *b* and 22.3*b*; for an example of the latter, see Plate III on page 404.

In § 22.4 we consider certain honeycombs (or "solid tessellations," or "degenerate polytopes") consisting of infinitely many solid cells in the same 3-space. In § 22.5 we see how these ideas help to explain some experimental results on the packing of equal spheres.

The geometry of this chapter is Euclidean. But all the other kinds of geometry can similarly be extended to spaces of any number of dimensions. As L. Fejes Tóth remarks in one of his books, we are able "to create an infinite set of new universes, the laws of which are within our reach, though we can never set foot in them."

22.1 THE SIMPLEST FOUR-DIMENSIONAL FIGURES

That ye . . . may be able to comprehend with all saints what is the breadth, and length, and depth, and height.

Ephesians III, 17-18

Spirits have four dimensions.

Henry More (1614-1687)

If an inhabitant of flatland was able to move in three dimensions, he would be credited with supernatural powers by those who were unable so to move; for he could appear or disappear at will, could (so far as they could tell) create matter or destroy it. . . . We may go one step lower, and conceive of a world of one dimension—like a long tube—in which the inhabitants could move only forwards and backwards. . . . Life in line-land would seem somewhat dull. . . . An inhabitant could know only two other individuals; namely, his neighbours, one on each side.

W. W. Rouse Ball

(*Mathematical Recreations and Essays*, 9th edition, 1920, p. 426)

When trying to appreciate the idea of Euclidean 4-space, we are helped by imagining the efforts of a hypothetical two-dimensional being to visualize a three-dimensional world.* In solid geometry we can find a line ("the third dimension") which is perpendicular to both of two intersecting lines and consequently perpendicular to every line in their plane. Analogously, in 4-space we can find a line ("the fourth dimension") which is perpendicular to all three edges of a trihedral solid angle, such as a corner of a cube, and consequently perpendicular to every line in the 3-space that contains the solid angle. It follows that two 3-spaces that have a common point have a common plane, and the product of reflections in them is a rotation about this plane, analogous to the familiar rotation about a line in three dimensions or about a point in two dimensions.

After accepting the idea of a fourth dimension, we can soon imagine a pyramid or a prism whose "base" is a solid. For instance, a regular tetrahedron $ABCD$ may serve as the base of a pyramid $ABCDE$ (Figure 22.1a) whose apex E is along the fourth dimension through the center of $ABCD$. If E is so chosen that its distances from A, B, C, D are all equal to the edge AB, we have a *regular simplex*, which may be regarded in five ways as a pyramid, each vertex in turn serving as the apex while the remaining four form the base.

Figure 22.1b is merely an octagon with a square drawn inwards on each

* *Flatland: A Romance of Many Dimensions*, by A Square (E. A. Abbott), Boston, 1885 and 1928.

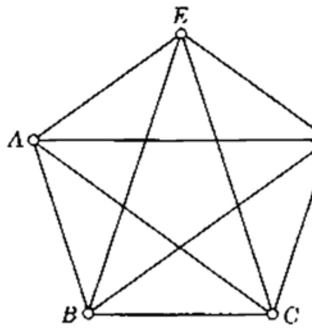


Figure 22.1a

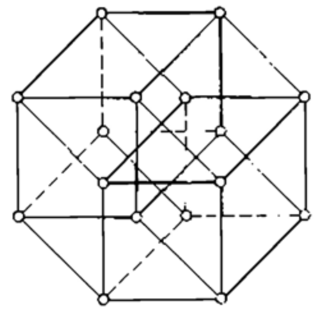


Figure 22.1b

side, or an octagon and an octagram with corresponding sides joined by squares. It may be regarded as a picture of a *hypercube* (or "8-cell," or "tessaract," or "measure polytope") which is a prism whose base is a cube, the "height" of the prism being equal to the edge of the base. Just as a cube can be traced out by moving a square along the third dimension, so a hypercube can be traced out by moving a cube along the fourth dimension. In Figure 22.1*b* the initial and final positions of the moving cube have been drawn in heavy lines. There are altogether eight cubes: these two, and six others traced out by the six faces. Each of the 24 squares (which appear in the figure as either squares or rhombi) belongs to two of the cubes, not lying in the same 3-space but rotated about the plane of the square until the two 3-spaces are at right angles.

The regular simplex and the hypercube are the two simplest instances

$$\{3, 3, 3\}, \qquad \{4, 3, 3\}$$

of a *regular polytope* $\{p, q, r\}$, which is a configuration of equal Platonic solids $\{p, q\}$, called *cells*, fitting together in such a way that each face $\{p\}$ belongs to two cells, and each edge to r cells. It follows that the arrangement of the cells at a vertex corresponds to the arrangement of the faces of a $\{q, r\}$, in the sense that each face of the $\{q, r\}$ is a vertex figure of the corresponding cell. This $\{q, r\}$, whose vertices are the midpoints of the edges at one vertex of $\{p, q, r\}$, is naturally called the *vertex figure* of the polytope. In fact, the three-digit Schläfli symbol $\{p, q, r\}$ is derived by "telescoping" the two-digit symbols $\{p, q\}$ and $\{q, r\}$ which denote the cell and the vertex figure.

We can now complete the first two rows of Table IV (on p. 414), in which the numbers of vertices, edges, faces, and cells are denoted by N_0, N_1, N_2, N_3. Although there is no easy formula for any of these numerical properties as a function of p, q, r, we can readily find their mutual ratios by arguments analogous to those that led to 10.31. In fact, if V, E, F refer to the cell $\{p, q\}$, and V', E', F' to the vertex figure $\{q, r\}$, we have

$$FN_3 = 2N_2, \quad VN_3 = F'N_0, \quad V'N_0 = 2N_1, \quad EN_3 = rN_1 = pN_2 = E'N_0.$$

For instance, the first equation comes from the observation that the F faces

of the N_3 cells are just the N_2 faces, counted twice because each belongs to two cells.

EXERCISES

1. The numbers $N_0 : N_1 : N_2 : N_3$ are proportional to

$$\frac{1}{q} + \frac{1}{r} - \frac{1}{2} \qquad \frac{1}{r} \qquad \frac{1}{p} \qquad \frac{1}{p} + \frac{1}{q} - \frac{1}{2}.$$

Thus $\{p, q, r\}$ satisfies Schläfli's four-dimensional analogue of Euler's theorem:

$$N_0 - N_1 + N_2 - N_3 = 0.$$

2. A hypercube of edge 1, with one vertex at the origin and 4 edges along the Cartesian axes, has the 16 vertices (x_1, x_2, x_3, x_4), where each of the four x's is either 0 or 1, independently.

3. A hypercube of edge 2, with its center at the origin and its edges parallel to the Cartesian axes, has the 16 vertices

$$(\pm 1, \pm 1, \pm 1, \pm 1).$$

4. Where is the center of the dilatation that relates the hypercubes described in the two preceding exercises?

22.2 A NECESSARY CONDITION FOR THE EXISTENCE OF {p, q, r}

". . . Space . . . is spoken of as having three dimensions, which one may call Length, Breadth, and Thickness. . . . But some philosophical people have been asking why three dimensions particularly—why not another direction at right angles to the other three? . . . I do not mind telling you I have been at work upon this geometry of Four Dimensions for some time. . . ."

H. G. Wells

(*The Time Machine*, 1895, p. 5)

It was apparently Kepler who first thought of the regular tessellations (§ 4.6) as infinite polyhedra. Analogously, the three-dimensional honeycomb of cubes (whose vertices may be taken to be all the points (x, y, z) for which x, y, z are integers) is the infinite polytope $\{4, 3, 4\}$: its cell is the cube $\{4, 3\}$, and its vertex figure is the octahedron $\{3, 4\}$ whose eight faces are the vertex figures of the eight cubes that surround a vertex, one in each "octant." The final 4 in the symbol $\{4, 3, 4\}$ means that there are four cells surrounding an edge. These four cubes fit together without any interstices because the dihedral angle of the cube is exactly a right angle. On the other hand, the hypercube $\{4, 3, 3\}$ is a finite polytope because the total angle at an edge is only three right angles, allowing the cells to be rotated out of the 3-space the way one derives a polyhedron by folding up its net (only now the angular deficiency is not related in any simple way to the number of vertices).

Similarly, since the dihedral angle of the tetrahedron $\{3, 3\}$ is slightly less than 71° (see Table II), we may place three, four, or five (but no more) tetrahedra together at a common edge, so as to begin the construction of $\{3, 3, 3\}$, $\{3, 3, 4\}$, or $\{3, 3, 5\}$. Again, since the dihedral angles of the octahedron and dodecahedron are between 90° and 120°, we may place just three of either together at an edge to obtain $\{3, 4, 3\}$ and $\{5, 3, 3\}$. But the icosahedron cannot be used in this manner, as its dihedral angle is greater than 120°. We have thus proved that the only possible finite regular polytopes in four dimensions are

$$\{3, 3, 3\}, \quad \{3, 3, 4\}, \quad \{3, 3, 5\}, \quad \{4, 3, 3\}, \quad \{3, 4, 3\}, \quad \{5, 3, 3\}.$$

The condition for $\{p, q, r\}$ to be a finite polytope may be expressed in general terms by recalling (from 10.43) that the dihedral angle of the Platonic solid $\{p, q\}$ is

$$2 \arcsin \left(\cos \frac{\pi}{q} \Big/ \sin \frac{\pi}{p} \right).$$

If r such angles together make less than 2π, each must be less than $2\pi/r$. Hence

$$\arcsin \left(\cos \frac{\pi}{q} \Big/ \sin \frac{\pi}{p} \right) < \frac{\pi}{r},$$

that is,

22.21 $$\cos \frac{\pi}{q} < \sin \frac{\pi}{p} \sin \frac{\pi}{r}.$$

Similarly, the condition for $\{p, q, r\}$ to be an infinite honeycomb filling three-dimensional space is

22.22 $$\cos \frac{\pi}{q} = \sin \frac{\pi}{p} \sin \frac{\pi}{r}:$$

an equation for which the only solution in integers greater than 2 is $\{4, 3, 4\}$.

EXERCISES

1. The condition 22.21 implies both 10.33 and the analogous inequality with p replaced by r. *Hint:*

$$\sin \frac{\pi}{p} \sin \frac{\pi}{r} < \sin \frac{\pi}{p}.$$

2. Obtain the Schläfli symbol for the regular polytope whose eight vertices are

$$(\pm 1, 0, 0, 0), \quad (0, \pm 1, 0, 0), \quad (0, 0, \pm 1, 0), \quad (0, 0, 0, \pm 1),$$

that is, for the polytope

$$|x_1| + |x_2| + |x_3| + |x_4| \leqslant 1.$$

22.3 CONSTRUCTIONS FOR REGULAR POLYTOPES

Though analogy is often misleading, it is the least misleading thing we have.

Samuel Butler (1835-1902)
(*Music, Pictures, and Books*)

We have seen that the inequality 22.21 is a *necessary* condition for the existence of a finite polytope $\{p, q, r\}$. The sufficiency of the condition requires an actual construction for each of the six figures. We know that r cells can fit together at an edge, but it is not obvious that the addition of further cells will ultimately yield a closed configuration in which every face of every cell belongs also to another cell.

As the complete story of such constructions is very long [Coxeter **1**, pp. 145–153], we must be content with a brief sketch, aided by analogy with what happens in three dimensions.

We recall that alternate vertices of a cube $\{4, 3\}$ belong to an inscribed tetrahedron $\{3, 3\}$ whose four faces correspond in an obvious manner to the four omitted vertices of the cube, whereas its six edges are diagonals of the six faces of the cube (*one* diagonal of each face). Moreover, the midpoints of these six edges, being the centers of the faces of the cube, are the vertices of an octahedron $\{3, 4\}$.

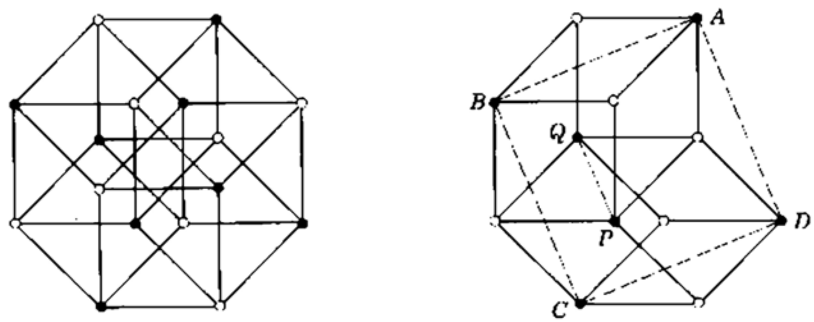


Figure 22.3a

Analogously, by selecting alternate vertices of the hypercube $\{4, 3, 3\}$ we obtain a polytope which has 8 vertices (the black points in Figure 22.3*a*) and 16 cells: one tetrahedron (such as $BCPQ$) corresponding to each of the 8 omitted vertices, and another (such as $ABPQ$) inscribed in each of the 8 cubic cells. This "16-cell" has 24 edges, which are diagonals of the 24 square faces of the hypercube (one diagonal of each face). Each of these 24 edges belongs to 4 tetrahedra (2 of each type, occurring alternately): for example, the edge PQ belongs to the 4 tetrahedra

$$ABPQ,\ BCPQ,\ CDPQ,\ DAPQ,$$

the first and third of which are inscribed in two adjacent cubes whose common face has PQ for one of its diagonals. We have thus proved that *the* 16-*cell is* $\{3, 3, 4\}$.

Completing the third line of Table IV, we observe that the numerical properties of $\{3, 3, 4\}$ are just those of $\{4, 3, 3\}$ in the reverse order. In fact, instead of obtaining the vertices of the 16-cell as alternate vertices of the hypercube, we could have obtained the vertices of another (similar) 16-cell as the centers of the cells of the hypercube. In other words, the hypercube and the 16-cell are *reciprocal* polytopes [Coxeter **1**, p. 127], like the cube and the octahedron. More generally, *the reciprocal of* $\{p, q, r\}$ *is* $\{r, q, p\}$.

The midpoints of the 24 edges of $\{3, 3, 4\}$ are the 24 vertices of a polytope whose cells are 24 octahedra: the vertex figures at the 8 vertices of $\{3, 3, 4\}$, and 16 inscribed in the 16 tetrahedra. Since all its cells are octahedra $\{3, 4\}$, this "24-cell" is $\{3, 4, 3\}$ [Hilbert and Cohn-Vossen **1**, p. 152, Fig. 172].

By suitably dividing the 12 edges of an octahedron in the ratio $\tau : 1$, we obtain the 12 vertices of an icosahedron (see § 11.2). By dividing the 96 edges of the 24-cell $\{3, 4, 3\}$ in this same ratio, we obtain the 96 vertices of a semiregular polytope $s\{3, 4, 3\}$ (the "snub 24-cell"), whose cells consist of 24 icosahedra and 120 tetrahedra: namely, at each vertex of the 24-cell, a set of 5 tetrahedra consisting of 1 surrounded by 4 others (like a partially folded "net" for the regular simplex $\{3, 3, 3\}$). When each icosahedral cell of $s\{3, 4, 3\}$ is capped by an icosahedral pyramid (the way an icosahedron is derived from a pentagonal antiprism by adding two pentagonal pyramids), we obtain a new polytope having a cluster of 20 tetrahedra to replace each of the 24 icosahedra, making a total of

$$24 \cdot 20 + 120 = 600$$

tetrahedra. The 120 vertices of this polytope consist of the 96 vertices of $s\{3, 4, 3\}$ and the 24 apices of the 24 icosahedral pyramids. (These 24 points, corresponding to the cells of the original $\{3, 4, 3\}$, are the vertices of a reciprocal $\{3, 4, 3\}$.) By careful examination [Coxeter **1**, pp. 152–153] we find that every edge belongs to 5 of the 600 tetrahedra. Hence the 600-cell is $\{3, 3, 5\}$ [Coxeter **1**, frontispiece].

Finally, the 120-cell $\{5, 3, 3\}$ (Figure 22.3*b* and Plate III) can be constructed as the reciprocal of $\{3, 3, 5\}$: its 600 vertices are the centers of the 600 tetrahedra. This information enables us to complete Table IV.

It is interesting to record that the snub 24-cell $s\{3, 4, 3\}$, which plays such a useful role in the above construction for $\{3, 3, 5\}$, was discovered by Thorold Gosset in 1897.* Figure 22.3*b* was drawn by B. L. Chilton. Plate III is a photograph of a wire model made by P. S. Donchian.

* Gosset was born in 1869 and died in 1962 [see Coxeter **1**, pp. 162–164].

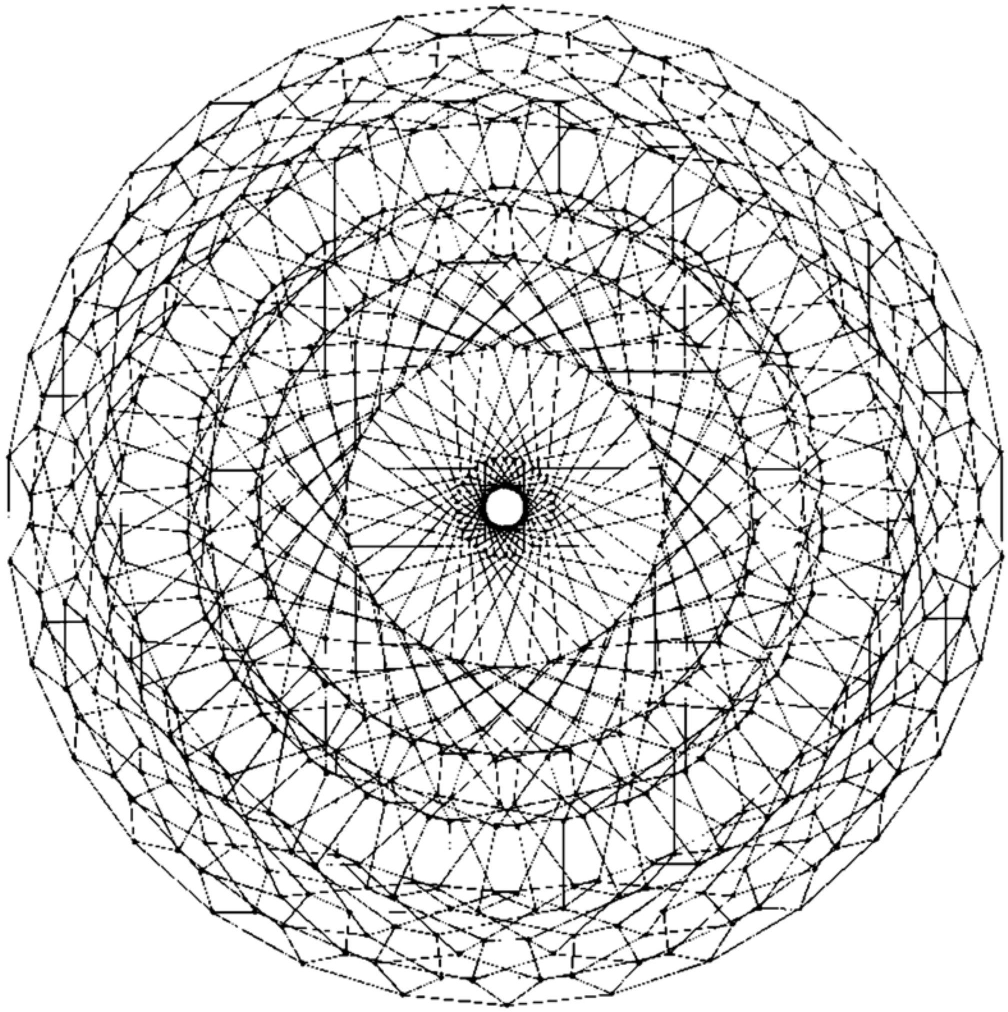


Figure 22.3b

Hartley [**1**, Nos. 56, 60] has given instructions for making models of a tetrahedron with an icosahedron placed on each face and of a dodecahedron with a dodecahedron placed on each face. When completed, these models show how we might begin to make solid nets for $s\{3, 4, 3\}$ and $\{5, 3, 3\}$, respectively.

EXERCISES

1. Locate the centers of the 8 cubic cells of the hypercube $(\pm 1, \pm 1, \pm 1, \pm 1)$.

2. Locate the midpoints of the 24 edges of the 16-cell

$$(\pm 2, 0, 0, 0), \quad (0, \pm 2, 0, 0), \quad (0, 0, \pm 2, 0), \quad (0, 0, 0, \pm 2).$$

3. Verify that the 96 vertices of $s\{3, 4, 3\}$, which are

$$(\pm\tau, \pm 1, \pm\tau^{-1}, 0),$$

evenly permuted, divide the 96 edges of the 24-cell $(\pm\tau, \pm\tau, 0, 0)$ (permuted) in the ratio $\tau : 1$.

4. The 120 vertices of the 600-cell $\{3, 3, 5\}$ are the 96 vertices of the above polytope $s\{3, 4, 3\}$, along with the 24 extra points

$$(\pm 2, 0, 0, 0) \quad \text{(permuted)} \quad \text{and} \quad (\pm 1, \pm 1, \pm 1, \pm 1).$$

5. The 600 vertices of the 120-cell $\{5, 3, 3\}$ are the permutations of

$$(\pm 2, \pm 2, 0, 0), \qquad (\pm\sqrt{5}, \pm 1, \pm 1, \pm 1),$$
$$(\pm\tau, \pm\tau, \pm\tau, \pm\tau^{-2}), \qquad (\pm\tau^2, \pm\tau^{-1}, \pm\tau^{-1}, \pm\tau^{-1})$$

along with the even permutations of

$$(\pm\tau^2, \pm\tau^{-2}, \pm 1, 0), \qquad (\pm\sqrt{5}, \pm\tau^{-1}, \pm\tau, 0), \qquad (\pm 2, \pm 1, \pm\tau, \pm\tau^{-1}).$$

(This corrects an error in the first edition of Coxeter **1**, p. 157.)

22.4 CLOSE PACKING OF EQUAL SPHERES

As the foot presses upon the sand when the falling tide leaves it firm, that portion of it immediately surrounding the foot becomes momentarily dry. . . . The pressure of the foot causes dilatation of the sand, and so more water is [drawn] through the interstices of the surrounding sand . . . , leaving it dry until a sufficient supply has been obtained from below, when it again becomes wet. On raising the foot we generally see that the sand under and around it becomes wet for a little time. This is because the sand contracts when the distorting forces are removed, and the excess of water escapes at the surface.

Osborne Reynolds (1842-1913)

(British Association Report, Aberdeen, 1885, p. 897).

Of all the two hundred thousand million men, women, and children who, from the beginning of the world, have ever walked on wet sand, how many, prior to the British Association Meeting at Aberdeen in 1885, if asked, "Is the sand compressed under your foot?" would have answered otherwise than "Yes!"? (Contrast with this the case of walking over a bed of wet sea-weed!)

Lord Kelvin (1824-1907)

(*Baltimore Lectures*, 1904, p. 625)

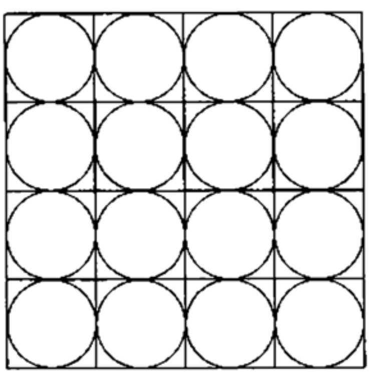


Figure 22.4a

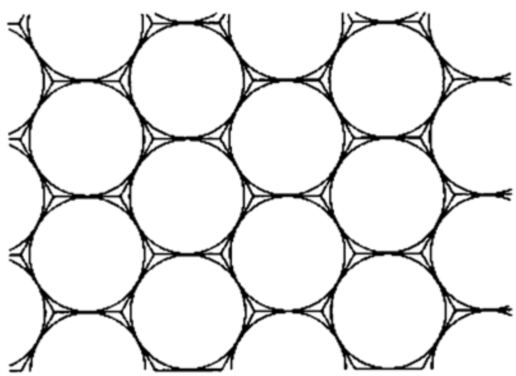


Figure 22.4b

Figures 22.4*a* and *b* show two possible ways of packing equal circles in a plane: the incircles of the faces of the regular tessellations $\{4, 4\}$ and $\{6, 3\}$ (§ 4.6). It is intuitively obvious that the latter is the more "economical" packing. To make this idea precise, we consider the incircles of the faces of the general regular tessellation $\{p, q\}$, and define the *density* of the packing to be the ratio of the area of a circle to the area of the $\{p\}$ in which it is inscribed. The density so defined is evidently less than 1, and the closest packing will have the greatest density, that is, the density nearest to 1. If

PLATE IV

face is a p-gon of side $2l$, its inradius is $r = l \cot \pi/p$ and its area is plr 2.91, 2.92); therefore, the density is

$$\frac{\pi r^2}{plr} = \frac{\pi}{p}\frac{r}{l} = \frac{\pi}{p}\cot\frac{\pi}{p} = \frac{\pi}{p}\Big/\tan\frac{\pi}{p}\,.$$

s is an increasing function of p, and tends to 1 when p tends to infinity. since the p-gon is a face of a regular tessellation, the only relevant val- of p are 3, 4, 6. Therefore the "best" value of p is 6, and the closest ılar packing consists of the incircles of the faces of $\{6, 3\}$, the density ıg

$$\frac{\pi}{6}\cot\frac{\pi}{6} = \frac{\pi}{6}\sqrt{3} = \frac{\pi}{2\sqrt{3}} = 0.9069\ldots$$

bert and Cohn-Vossen **1**, p. 47].

It can easily be proved that this is still the closest packing when we abandon the requirement of regularity but insist instead that the centers of the circles form a lattice [Hilbert and Cohn-Vossen **1**, pp. 33–35]. Actually, even this restriction can be abandoned [Darwin **1**, p. 345; Fejes Tóth **1**, p. 58], as the bees discovered millions of years ago (Plate IV).

An analogous packing of spheres in three-dimensional space may be obtained by taking the inspheres of the cells of a honeycomb of equal polyhedra. The density is naturally defined as the ratio of the volume of a sphere to the volume of the cell in which it is inscribed. In the case of $\{4, 3, 4\}$, the honeycomb of cubes of edge $2l$, this is

$$\frac{\frac{4}{3}\pi l^3}{(2l)^3} = \frac{\pi}{6} = 0.5236\ldots.$$

A greater density can be obtained by using the *midspheres* (§ 10.4) of *alternate* cells, as we shall soon see.

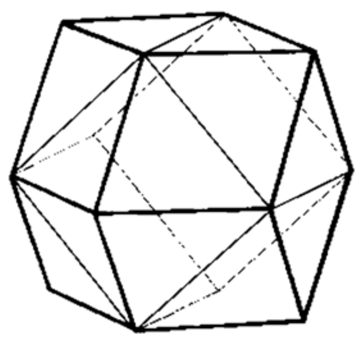


Figure 22.4c

If we imagine the cells of the cubic honeycomb to be colored alternately black and white, like a three-dimensional chessboard, we may dissect each white cube into six square pyramids (by planes joining pairs of opposite edges) and attach each pyramid to the neighboring black cube. Each black cube is now covered with six white pyramids, one on each face, to form a *rhombic dodecahedron* (Figure 22.4*c*), whose twelve rhombic faces have the twelve edges of the black cube for their shorter diagonals [Steinhaus **2**, p. 152]. Thus the insphere of the rhombic dodecahedron is the midsphere of the cube, of radius $\sqrt{2}\,l$, and the volume of the rhombic dodecahedron is twice that of the cube, namely, $2(2l)^3 = 16l^3$. In the honeycomb of such larger cells, each insphere is the midsphere of a black cube, and such spheres touch one another at the centers of the rhombic faces, that is, at the midpoints of the edges of the original honeycomb of cubes. Thus each sphere touches *twelve* others, the points of contact being the midpoints of the twelve edges of a cube. The density of this *cubic close packing* is evidently

$$\frac{\frac{4}{3}\pi(\sqrt{2}l)^3}{16l^3} = \frac{\pi}{3\sqrt{2}} = 0.74048\ldots$$

[Hilbert and Cohn-Vossen **1**, p. 47].

The rhombic dodecahedron occurs in nature as a crystal of garnet, and the three-dimensional chessboard occurs as the arrangement of atoms in a crystal of common salt, with a sodium atom in each black cube and a chlorine atom in each white cube (or vice versa). The centers of the black cubes, which are the centers of the spheres in cubic close packing, are easily seen to form the face-centered cubic lattice. It follows from § 18.4 that this is the densest possible packing of spheres whose centers form a lattice.

In old war memorials we often see a pyramidal pile of cannon balls: one at the apex resting on four others which, in turn, rest on nine, and so on. Each interior ball touches 12 others: 4 in its own layer, 4 above, and 4 below. In fact, these cannon balls are arranged in cubic close packing [Kepler **1**, pp. 268–269]. The base of the square pyramid consists of (say) n^2 balls arranged like the circles in Figure 22.4*a*. When n is large, the shape of the whole pyramid is essentially the "top" half of a regular octahedron (regarded as a square dipyramid); each sloping face is an equilateral triangle formed by $1 + 2 \ldots + n$ balls.

By turning the pyramid over so that such a sloping face becomes horizontal, we obtain a different aspect of the same packing. In this aspect we begin with a horizontal layer of spheres whose "equators" are the incircles of the hexagons of $\{6, 3\}$, as in Figure 22.4*b*. The next higher layer is just like this but shifted slightly to the right (say), so that each sphere rests on three, its center being vertically above a vertex of $\{6, 3\}$ from which an edge goes off to the right. Since all the centers form a three-dimensional lattice, the spheres in the third layer (resting on the second) are shifted again to the right, so that each center is vertically above a vertex of $\{6, 3\}$ from which an edge goes off to the left. The fourth layer is vertically above the first, and thereafter the sequence recurs.

In 1883, the crystallographer Barlow described an equally dense packing in which the centers do not form a lattice. This can be derived by taking the same horizontal layers in a different order. More precisely, we discard the "third layer" just described and substitute a new third layer vertically above the first. Then we add a fourth layer vertically over the second, and so on; the shifting from one layer to the next is alternately to the right and left, like a zigzag. This nonlattice packing is called *hexagonal close packing* [Ball **1**, p. 150; Hilbert and Cohn-Vossen **1**, p. 46; Steinhaus **2**, p. 170; Fejes Tóth **1**, pp. 172–173].

Helpful models to illustrate these ideas are provided by fourteen golf balls and two shallow trays of dimensions 5 in. $\times$ 5 in. and 4.6 in. $\times$ 5.8 in., respectively. Either tray will hold nine balls in three rows of three. In the square tray, a pyramidal "cannon ball" arrangement can be completed by adding four more above and the remaining one at the top. In the oblong tray, the four balls in the second layer should have their centers at the vertices of a rhombus, not a

rectangle. The third layer is again represented by just one ball, but now there are two possible positions for it: one belongs to cubic close packing and the other to hexagonal close packing.

Since hexagonal close packing has the same density as cubic close packing, namely 0.74048 . . . , it is natural to ask whether some still less systematic packing (without any straight rows of spheres) may have a greater density. This remains an open question. The best theoretical approach to an answer is the proof by Rogers* that, if such a packing exists, its density must be less than 0.7797. . . .

Experiments in this direction began as long ago as 1727, when Stephen Hales stated, in his *Vegetable Staticks,*

I compressed several fresh parcels of Pease in the same Pot, with a force equal to 1600, 800, and 400 pounds; in which Experiments, tho' the Pease dilated, yet they did not raise the lever, because what they increased in bulk was, by the great incumbent weight, pressed into the interstices of the Pease, which they adequately filled up, being thereby formed into pretty regular Dodecahedrons.

Hales presumably reached his conclusion by observing some pentagonal faces on his dilated peas. They could not all have been regular dodecahedra. For, since the dihedral angle of the regular dodecahedron is less than 120° (see Table II on p. 413), three such solids with a common edge will leave an angular gap of about 10° 19′. In fact, dodecahedra $\{5, 3\}$ are the cells of the configuration $\{5, 3, 3\}$, which is not an infinite three-dimensional honeycomb but a finite four-dimensional polytope.

In 1939, the botanists J. W. Marvin and E. B. Matzke repeated Hales's experiment, replacing his peas by lead shot, "carefully selected under a microscope for uniformity of size and shape," in a steel cylinder, compressed with a steel plunger at a sufficient pressure (40,000 pounds) to eliminate all interstices.† When the shot were stacked in cannon-ball fashion and compressed, they became nearly perfect rhombic dodecahedra. But "if the shot were just poured into the cylinder the way Hales presumably put his peas into the iron pot, irregular 14-faced bodies were formed." Almost all the faces were either quadrangles, pentagons, or hexagons, with pentagons predominating. Another botanist examined cells in undifferentiated vegetable tissues, and concluded that the internal cells have an average of approximately 14 faces, though the most prevalent shape (occurring 32 times among the 650 cells examined) had 13 faces: 3 quadrangles, 6 pentagons, and 4 hexagons. The few cells that had only 12 faces were neither rhombic dodecahedra nor regular dodecahedra.

Matzke also made a microscopic examination of a froth of 1900 measured bubbles. "For 600 central bubbles examined, the average number of contacts was 13.7." The commonest shape had again 13 faces: 1 quadrangle, 10 pentagons, and 2 hexagons.

* C. A. Rogers, The packing of equal spheres, *Proceedings of the London Mathematical Society* (3), **8** (1958), pp. 609–620.

† E. B. Matzke, In the twinkling of an eye, *Bulletin of the Torrey Botanical Club,* **77** (1950), pp. 222–227.

In 1959, Professor Bernal* confirmed the prevalence of pentagonal faces by a remarkably simple experiment in which equal balls of "Plasticene" (oily modeling clay) were rolled in powdered chalk, packed together irregularly, and pressed into one solid lump. The resulting polyhedra were found to have an average of 13.3 faces.

To test the possibility that a random packing of equal spheres might attain a density between 0.7405 and 0.7797, G. D. Scott poured thousands of ball bearings into spherical flasks of various sizes, gently shaking each flask as it was being filled. Assuming that the exceptional situation at the surface of the container will make the density

$$\rho - \varepsilon N^{-1/3}$$

for N balls, where ρ and ε are constants, he found from these experiments a closest random packing with

$$\rho = 0.6366, \qquad \varepsilon = 0.33.$$

By careful filling of the flasks without shaking, a loosest incompressible random packing was found with

$$\rho = 0.60, \qquad \varepsilon = 0.37.$$

Since, for the closest random packing, ρ falls far short of 0.7405, it seems unlikely that any greater density can be maintained throughout a region that extends indefinitely in all directions.

If we could fill a spherical flask with N ball bearings in cubic close packing, we would expect the density to be expressible as a series beginning with the two terms

$$0.7405 - \varepsilon N^{-1/3}.$$

But this experiment does not seem to be feasible. A scholarly book has been written on the theory of lattice points in spheres† without throwing any light on the value of ε in this 3-dimensional case, although considerable progress has been made on the analogous problem in spaces of other numbers of dimensions, such as 2 or 4.

Whatever the closest random packing may be, it is clear from Osborne Reynolds's experiment on the seashore that any small disturbance increases the size of the interstices. The same principle may explain a Hindu fakir's magic trick, which was mentioned by Martin Gardner. A cylindrical jar with a rather narrow opening at the top is filled with uncooked rice, gently shaken down so as to be well packed. A table knife is plunged repeatedly into the jar, to a greater depth each time. After about a dozen plunges, the knife will suddenly bind so that, when raised by the handle, it will support the whole jar of rice.

* J. D. Bernal, A geometrical approach to the structure of liquids, *Nature*, 183 (1959), pp. 141–147.

† Arnold Walfisz, *Gitterpunkte in mehrdimensionalen Kugeln*, Warsaw, 1957.

EXERCISES

1. Is the arrangement of incircles of all the faces of the tessellation $\{4, 4\}$ (Figure 22.4*a*) any less dense than that of the circumcircles of alternate faces, i.e., the circumcircles of the black squares of a chessboard?

2. Is it possible to arrange seven equal non-overlapping spheres in such a way that two of them touch each other and both touch all the remaining five, while these five (a) form a ring in which each touches two others? (b) do not touch one another at all?

3. Is it possible to arrange thirteen equal non-overlapping spheres in such a way that one of them touches all the remaining twelve while these twelve do not touch one another at all?

4. A pyramidal pile with n layers contains $n(n + 1)(2n + 1)/6$ cannon balls [Ball **1**, p. 59]; a tetrahedral pile contains $n(n + 1)(n + 2)/6$. In both cases the arrangement is cubic close packing.

22.5 A STATISTICAL HONEYCOMB

The fluidity of a liquid is a consequence of its molecular irregularity.

J. D. Bernal (1901 -)

Three equal circles in a plane are packed as closely as possible when they all touch one another. The two-dimensional problem of close packing is easy because any number of further circles can be added in such a way as to continue the pattern systematically over the whole plane. This is, as we have seen, the pattern formed by the incircles of the faces of the regular tessellation of hexagons, $\{6, 3\}$ (Figure 22.4*b*).

Analogously in space, four equal spheres are packed as closely as possible when they all touch one another, and some further spheres can be added so as to form the beginning of a pattern apparently consisting of the inspheres of the cells of a regular honeycomb $\{p, 3, 3\}$. Although the equation 22.22 has no integral solution when $q = r = 3$, we naturally conclude that a compressed random packing of equal lead shot, a nearly homogeneous aggregate of vegetable cells, and a froth of equal bubbles, are all somehow trying to approximate to a honeycomb $\{p, 3, 3\}$ in which p lies between 5 and 6. The fractional value of p means that this "honeycomb" can exist only in a statistical sense, but the agreement with experiment is striking.

When $q = r = 3$, the equation 22.22 actually becomes

22.51 $$\sin\frac{\pi}{p} = \cot\frac{\pi}{3} = \sqrt{\frac{1}{3}}.$$

This shows that the angle $180°/p$ is $35°\ 15'\ 52''$, which is half the dihedral angle of the regular tetrahedron $\{3, 3\}$ (see Table II). (In fact, we may regard p as the number of regular tetrahedra $\{3, 3\}$ that can be placed together around a common edge, as if we were beginning to construct the dual honeycomb $\{3, 3, p\}$ whose vertices are the centers of the spheres.) Thus

$$p = \frac{180}{35.264\ldots} = 5.1044\ldots,$$

in agreement with Matzke's observation that pentagons are prevalent (especially in a froth) whereas hexagons are more frequent than quadrangles. The cell $\{p, 3\}$ has an average of F faces and V vertices where, by 10.32 with $q = 3$,

$$F = \frac{12}{6 - p} = 13.398\ldots, \qquad V = \frac{4}{(6/p) - 1} = 22.796\ldots,$$

in reasonably close agreement with Matzke's 13.7, with Bernal's 13.3, and with one of the two theoretical models proposed by Meijering,* who used intricate statistical methods to obtain $V = 22.56\ldots$. A fourth theoretical model [Coxeter **4**, p. 756] yields

$$F = \tfrac{1}{3}(23 + \sqrt{313}) = 13.564\ldots, \qquad V = \tfrac{2}{3}(17 + \sqrt{313}) = 23.128\ldots.$$

EXERCISE

In the "twisted prism" formed by 28 regular tetrahedra

$$A_0A_1A_2A_3, \quad A_1A_2A_3A_4, \quad \ldots, \quad A_{27}A_{28}A_{29}A_{30},$$

the broken line $A_0A_3A_6A_9 \ldots A_{30}$ consists of 10 equal chords of a circular helix. Taking the axis of this helix to be vertical, do we find the vertex A_{30} exactly above A_0? (A model can be conveniently made by fastening together 87 equal sticks from the "D-stix Pre-engineering Kit 701," manufactured in Yardley, Wash.) For the whole story, we regard the tetrahedra as being 28 cells of the "honeycomb" $\{3, 3, p\}$, where p is given by 22.51. ($A_0A_1A_2 \ldots$ is a "Petrie polygon" of this honeycomb.) Setting

$$\cos^2 \frac{\pi}{p} = \frac{2}{3}$$

and $q = r = 3$ in the equation 12.35 of Coxeter [**1**, p. 221], we obtain $\xi_1 = 0$ and $\cos \xi_2 = -\tfrac{2}{3}$. The angle between the planes joining the axis to A_0 and A_{30} is

$$30(\xi_2 - 120°) = 354°\,20'.$$

(This ξ_2, being nearly $131°\,49'$, is remarkably close to the corresponding property of the four-dimensional polytope $\{3, 3, 5\}$, which is exactly 132° [Coxeter **1**, p. 247].)

* J. L. Meijering, *Philips Research Reports*, **8** (1953), p. 282. The value $V = 22.79\ldots$ was first obtained by C. S. Smith, *Acta Metallurgica*, **1** (1953), p. 299. See also E. N. Gilbert, *Annals of Mathematical Statistics*, **33** (1962), pp. 958–972, and R. E. Williams, *Science*, **161** (1968), pp. 276–277.

Table I
The 17 Space Groups of Two-Dimensional Crystallography (§ 4.3)

Symbol	Generators
p1	Two translations
p2	Three half-turns
pm	Two reflections and a translation
pg	Two parallel glide reflections
cm	A reflection and a parallel glide reflection
pmm	Reflections in the four sides of a rectangle
pmg	A reflection and two half-turns
pgg	Two perpendicular glide reflections
cmm	Two perpendicular reflections and a half-turn
p4	A half-turn and a quarter-turn
p4m	Reflections in the three sides of a (45°, 45°, 90°) triangle
p4g	A reflection and a quarter-turn
p3	Two rotations through 120°
p3m1	A reflection and a rotation through 120°
p31m	Reflections in the three sides of an equilateral triangle
p6	A half-turn and a rotation through 120°
p6m	Reflections in the three sides of a (30°, 60°, 90°) triangle

Table II
The Five Platonic Solids (§ 10.3)

Name	Schläfli Symbol	V	E	F	Dihedral Angle
Tetrahedron	$\{3, 3\}$	4	6	4	70° 32′ −
Cube	$\{4, 3\}$	8	12	6	90°
Octahedron	$\{3, 4\}$	6	12	8	109° 28′ +
Dodecahedron	$\{5, 3\}$	20	30	12	116° 34′ −
Icosahedron	$\{3, 5\}$	12	30	20	138° 11′ +

Table III
The Finite Groups of Isometries (§ 15.5)

Rotation Groups			Direct Products		Mixed Groups	
Name	Symbol	Order	Symbol	Order	Symbol	Order
Cyclic	C_n	n	$C_n \times \{I\}$	$2n$	$C_{2n}C_n$	$2n$
Dihedral	D_n	$2n$	$D_n \times \{I\}$	$4n$	D_nC_n	$2n$
Tetrahedral	A_4	12	$A_4 \times \{I\}$	24	$D_{2n}D_n$	$4n$
Octahedral	S_4	24	$S_4 \times \{I\}$	48	S_4A_4	24
Icosahedral	A_5	60	$A_5 \times \{I\}$	120		

Table IV
The Regular Polytopes {p, q, r} (§ 22.2)

Name	Schläfli Symbol	N_0	N_1	N_2	N_3
Regular simplex	{3, 3, 3}	5	10	10	5
Hypercube	{4, 3, 3}	16	32	24	8
16-cell	{3, 3, 4}	8	24	32	16
24-cell	{3, 4, 3}	24	96	96	24
120-cell	{5, 3, 3}	600	1200	720	120
600-cell	{3, 3, 5}	120	720	1200	600
Cubic honeycomb	{4, 3, 4}	∞	∞	∞	∞

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Answers to Exercises

§1.3

1. The reflection in the line $x = y$ interchanges x and y.

2. If the line BA meets the circle in P (beyond A) and P' (between A and B), we have $BC^2 = BP \times BP' = (BA + AC)(BA - AC)$.

3. The triangle CDF is equilateral; so is ABC.

4. This result was conjectured by Paul Erdös and first proved by L. J. Mordell (see the *American Mathematical Monthly*, **44** (1937), p. 252, problem 3740, or Fejes Tóth **1**, pp. 12–14). In 1960, Mordell discovered the following simpler proof. For convenience, let OA, OB, OC, OP, OQ, OR be denoted by x, y, z, p, q, r, so that the theorem to be proved is

$$x + y + z \geqslant 2(p + q + r).$$

Also let p', q', r' denote the lengths (within the triangle) of the bisectors of the angles

$$2\alpha = \angle BOC, \qquad 2\beta = \angle COA, \qquad 2\gamma = \angle AOB.$$

By comparing the area of the triangle OBC with the two parts into which it is dissected by the bisector p', and using the well-known inequality $y + z \geqslant 2\sqrt{yz}$ which comes from $(\sqrt{y} - \sqrt{z})^2 \geqslant 0$, we find

$$yz \sin 2\alpha = p'(y + z) \sin \alpha \geqslant 2p'\sqrt{yz} \sin \alpha,$$

with equality only when $y = z$. Hence $\sqrt{yz} \cos \alpha \geqslant p'$, and similarly $\sqrt{zx} \cos \beta \geqslant q'$, $\sqrt{xy} \cos \gamma \geqslant r'$. Since

$$\begin{aligned} x + y + z - 2\sqrt{yz}\cos\alpha - 2\sqrt{zx}\cos\beta - 2\sqrt{xy}\cos\gamma \\ = (\sqrt{x} - \sqrt{y}\cos\gamma - \sqrt{z}\cos\beta)^2 + (\sqrt{y}\sin\gamma - \sqrt{z}\sin\beta)^2 \geqslant 0, \end{aligned}$$

it follows that

$$x + y + z \geqslant 2(p' + q' + r') \geqslant 2(p + q + r).$$

The inequality $x + y + z \geqslant 2(p' + q' + r')$, which may be regarded as an extended form of the Erdös-Mordell theorem, was first noticed by D. F. Barrow. See also O. Bottema, R. Ž. Djordjević, R. R. Janić, D. S. Mitrinović and P. M. Vasić, *Geometric Inequalities* (Wolters-Noordhoff, Groningen, The Netherlands, 1969), p. 139.

5. Equality occurs only if $x = y = z$ and $\sqrt{y} \sin \gamma - \sqrt{z} \sin \beta = 0$, etc., so that $\alpha = \beta = \gamma$. The triangle is equilateral, and O is its center.

6. Let h_a denote the "altitude" from A to BC, and Δ the area of the triangle ABC.

Since $x + p \geqslant h_a$, we have

$$a(x + p) \geqslant ah_a = 2\Delta = ap + bq + cr,$$

whence

$$ax \geqslant bq + cr.$$

Consideration of similar triangles shows that this remains true when the line AO is extended (even if O is on the far side of the base BC). Applying the same inequality to the image of O by reflection in the internal bisector of $\angle BAC$, we obtain

$$ax \geqslant br + cq.$$

Adding these two inequalities for ax, we obtain

$$2ax \geqslant (b + c)(q + r).$$

Multiplying together this and two other inequalities of the same kind,

$$8\, abcxyz \geqslant (b + c)(c + a)(a + b)(q + r)(r + p)(p + q).$$

Since $b + c \geqslant 2\sqrt{bc}$, etc., we have $(b + c)(c + a)(a + b) \geqslant 8abc$. Hence

$$xyz \geqslant (q + r)(r + p)(p + q).$$

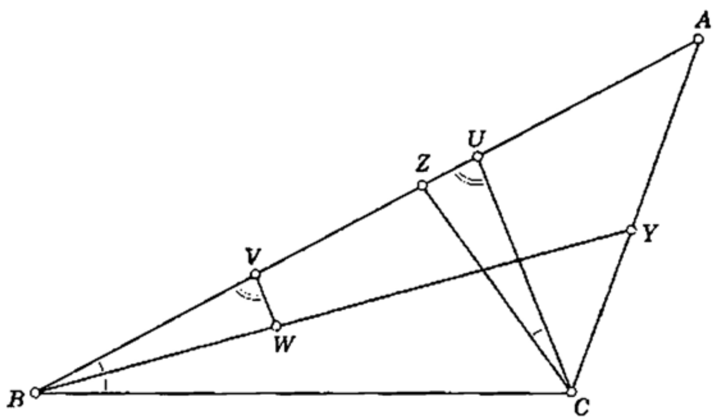


Figure 1-3e

7. Let B be the smaller of the two different angles B and C of the triangle ABC. Let BY and CZ be the internal bisectors of these angles, as in Figure 1.3e. Take U on AZ so that $\angle ZCU = \frac{1}{2}B$. Since the triangle UBC has a smaller angle at B than at C, $BU > CU$. Take V on BU so that $BV = CU$. Take W on BY so that $\angle BVW = \angle CUZ$. By the angle-side-angle criterion, BVW and CUZ are congruent triangles, and $BW = CZ$. But W and Y are on opposite sides of the line CU. Hence $BY > BW$, that is, $BY > CZ$.

This theorem was proposed in 1840 by C. L. Lehmus, and proved by Jacob Steiner. For its history, see J. A. McBride, *Edinburgh Mathematical Notes*, **33** (1943), pp. 1–13. McBride asserts that more than sixty proofs have been given. The simple one given above came in a letter from H. G. Forder; it excels most by being "absolute" in the sense of §12.1. For the original proofs by Steiner and Lehmus, respectively, see *Journal für die reine und angewandte Mathematik*, **28** (1844), p. 376, and *Archiv der Mathematik und Physik*, **15** (1850), p. 225.

§1.4

1. Apply Euclid I.5 to the isosceles triangle GBC and then I.4 to the two triangles $B'BC$, $C'CB$.

2. Add together three inequalities such as

$$\tfrac{2}{3}BB' + \tfrac{2}{3}CC' > BC.$$

Complete the parallelogram $CABK$ and observe that twice the median from A is $AK < AC + CK = b + c$.

§1.5

1. The circle through P with center O.

2. Use 1.52.

3. Let the tangents to the incircle from A, B, C be t_a, t_b, t_c. Then

$$t_b + t_c = a, \qquad t_c + t_a = b, \qquad t_a + t_b = c;$$

therefore

$$t_a = \tfrac{1}{2}(b + c - a) = s - a.$$

4. By Euclid III.20, if the angle at the circumference is greater than 90°, the angle at the center is greater than 180°.

5. At the midpoint of the hypotenuse.

6. Make repeated use of Pythagoras's theorem.

7. Since $bc + ca + ab = \{(s - a)(s - b)(s - c) + abc + s^3\}/s = r^2 + 4Rr + s^2$ and $abc = 4R\Delta = 4Rrs$, we have

$$\begin{aligned}(2R - a)(2R - b)(2R - c) &= 8R^3 - 8R^2s + 2R(r^2 + 4Rr + s^2) - 4Rrs\\ &= 2R(2R + r - s)^2.\end{aligned}$$

Alternatively, in the notation of ex. 3,

$$\begin{aligned}(t_a - r)(t_b - r)(t_c - r) &= (s - r - a)(s - r - b)(s - r - c)\\ &= (s - r)^3 - 2s(s - r)^2 + (r^2 + 4Rr + s^2)(s - r) - 4Rrs\\ &= 2r^2(s - r - 2R).\end{aligned}$$

This yields the desired criterion, since the angle A is acute or right or obtuse according as $t_a - r$ is positive or zero or negative. In fact, we can conclude further that *the triangle has an obtuse angle if and only if* $r + 2R > s$. Corrado Ciamberlini [*Bolletino della Unione Matematica Italiana* (2), **5** (1943), pp. 37–41] observed that

$$4R^2 \cos A \cos B \cos C = s^2 - (r + 2R)^2.$$

8. Since $2\eta_1 = -\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4$, $\epsilon_1 + \eta_1 = \tfrac{1}{2}\Sigma\epsilon_i$. Also $\Sigma\epsilon_i\eta_i = \Sigma\epsilon_i(\epsilon_i + \eta_i) - \Sigma\epsilon_i^2 = \tfrac{1}{2}(\Sigma\epsilon_i)^2 - \Sigma\epsilon_i^2 = 0$.

9. If $\epsilon_1 = \epsilon_2 = 0$, Beecroft's equations imply $\eta_3 = \eta_4 = 0$ and $\epsilon_3 = \epsilon_4 = \eta_1 = \eta_2$, so that the configuration consists of four lines E_1, H_3, E_2, H_4 forming a square, and four circles H_2, E_4, H_1, E_3 having the sides of the square for diameters. This is what happens when $k + l = m + n = 0$. In any other case, we can assign arbitrary values to the three bends ϵ_1, ϵ_2, η_3, subject only to the condition $\epsilon_1 + \epsilon_2 > 0$ (which ensures that if E_1 and E_2 have internal contact, the larger circle is the one whose bend is negative). Then η_4 is determined by the simple equation

$$\epsilon_1 + \epsilon_2 = \eta_3 + \eta_4$$

(which follows from $\epsilon_1 + \epsilon_2 - \epsilon_3 + \epsilon_4 = 2\eta_3$, $\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4 = 2\eta_4$), and the remaining bends are

$$\epsilon_3 = \frac{\eta_4^2 - \epsilon_1\epsilon_2}{\epsilon_1 + \epsilon_2}, \qquad \epsilon_4 = \frac{\eta_3^2 - \epsilon_1\epsilon_2}{\epsilon_1 + \epsilon_2}, \qquad \eta_1 = \frac{\epsilon_2^2 - \eta_3\eta_4}{\epsilon_1 + \epsilon_2}, \qquad \eta_2 = \frac{\epsilon_1^2 - \eta_3\eta_4}{\epsilon_1 + \epsilon_2}.$$

The proposed parametrization is obtained by choosing

$$k = \epsilon_1/\sqrt{\epsilon_1 + \epsilon_2}, \qquad l = \epsilon_2/\sqrt{\epsilon_1 + \epsilon_2},$$
$$m = -\eta_3/\sqrt{\epsilon_1 + \epsilon_2}, \qquad n = -\eta_4/\sqrt{\epsilon_1 + \epsilon_2}.$$

10. Using 1.59, 1.58, 1.52, and 1.56 in turn, we obtain

$$\epsilon_4 = \epsilon_1 + \epsilon_2 + \epsilon_3 - 2\eta_4 = \frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} \pm \frac{2}{r}$$
$$= \frac{r_a + r_b + r_c \pm 2s}{\Delta} = \frac{r + 4R \pm 2s}{\Delta}.$$

These two circles (which touch three mutually tangent circles) became known as *Soddy's circles* before anyone noticed the earlier work of Descartes and Steiner [**1**, pp. 60–63, 524]. As Descartes used the letters d, e, f, x for the radii of four mutually tangent circles, he undoubtedly saw that he had obtained a quadratic equation for x in terms of d, e, f.

11. Let $A_1B_1C_1$ be the feet of the perpendiculars from P to BC, CA, AB. Applying Euclid III.21 or 22 to each of the cyclic quadrangles PA_1B_1C, $PABC$, PA_1BC_1, show that $\angle PA_1B_1$ and $\angle PA_1C_1$ are either equal or supplementary.

12. $\angle C_3B_3A_3 = \angle C_3P_3P + \angle PB_3A_3 = \angle CBP + \angle PBA = \angle CBA$.

§1.6

1. The circumcenter of the new triangle is the orthocenter of ABC.

3. At the vertex where the right angle occurs.

4. Because $\sin B = \sin C$.

5. On $B'E$, take C so that $GC = GB$, and A so that $AB' = B'C$.

6. It is $b \sin C$, and $b = 2R \sin B$.

7. One-third of the altitude.

8. If the Euler line passes through A, and if A is not a right angle, the altitude line AH is a median.

9. $R \cos A = \frac{2}{3}R \sin B \sin C$.

§1.7

1. (a) One pair. (b) Three pairs.

2. If $A > B > C$, the order is $EA''FC'B''DA'C''B'$.

3. The angular measures of the relevant arcs of the nine-point circle (with $A > B > C$, as in Figure 1.7*a*) are

$$A'E = A'F = 2A, \qquad B'F = B'D = 2B, \qquad C'D = C'E = 2C,$$

whence

$$DA' = 2(\pi - 2C - A) \qquad A'B' = 2(\pi - A - B),$$
$$B'E = 2(2A + B - \pi), \qquad EF = 2(\pi - 2A), \qquad FC'' = 2(2A + C - \pi).$$

4. The internal and external bisectors of an angle are perpendicular.

5. The nine-point center of $I_aI_bI_c$ is the circumcenter of its orthic triangle ABC.

6. Each circumradius is twice the radius of the nine-point circle.

§1.8

1. $U'V$, which passes through W, is the image of UV by reflection in AC.

2. When there is a right angle at A, V and W coincide with A. When there is an obtuse angle at A, this degenerate triangle UAA is still "better" than any proper triangle.

3. The lines joining pairs of centers, being perpendicular to the common chords AP, BP, CP, make angles of 60° with one another, and thus form an equilateral triangle.

4. At the Fermat point. (H. G. Forder uses analogous considerations to prove that, if $ABCD$ is a tetrahedron and $PA + PB + PC + PD$ is a minimum, the angles APB and CPD are equal and their bisectors lie on one line.)

5. They join pairs of villages to the ends of a short road in the middle.

6. The "best" point for the "very obtuse" triangle is A itself. For the convex quadrangle it is the point of intersection of the diagonals.

7. P is the incenter of $P'BC$.

8. Let Z, X, U be the centers of the squares on three consecutive sides AB, BC, CD of the parallelogram $ABCD$. The triangle XBZ is derived from XCU by a quarter-turn (i.e., rotation through a right angle) about X.

9. Let M be the midpoint of CA. By ex. 8, the segments MZ and MX are congruent and perpendicular. The same can obviously be said of MY and MA. Therefore the triangle MAX is derived from MYZ by a quarter-turn about M.

10. As in ex. 9, the segments MZ and MX are congruent and perpendicular. Similarly (by considering the triangle CDA instead of ABC), the segments MU and MV are congruent and perpendicular. Therefore the triangle MXV is derived from MZU by a quarter-turn about M.

§1.9

1. These lines are the medians of the equilateral triangle PQR.

2. (i) $\alpha = \beta = \gamma = 40°$; (ii) $\alpha = 30°$, $\beta = \gamma = 45°$.

3. Since $\angle CP_1Q = \angle CPQ = \gamma + \alpha = \angle QRA$, the circumcircle of AQR passes through P_1, and likewise through P_2. Since

$$P_1Q = PQ = QR = RP = RP_2,$$

the points P_1, Q, R, P_2 are evenly spaced along this circle. In the special case, each of the arcs P_1Q, QR, RP_2 subtends 20° at A, and the triangle AQR is isosceles.

§2.1

1. Since
$$\frac{ON_1}{N_1P_0} = \frac{OD}{DP_0} = \frac{1}{\sqrt{5}},$$
$$\frac{ON_1}{OP_1} = \frac{ON_1}{ON_1 + N_1P_0} = \frac{1}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{4} = \cos 72°.$$

2. $\frac{6}{17} - \frac{1}{3} = \frac{1}{51}$.

§2.2

If s is odd, $x^s + 1$ is divisible by $x + 1$, and therefore $2^{rs} + 1$ by $2^r + 1$.

§2.4

1. Suppose a given isometry of period 2 interchanges A and A', and interchanges B and B', where B does not lie on AA'. The midpoints of AA' and BB' are invariant

points. If they are distinct, the isometry is a reflection (by 2.31). If they coincide, it is a half-turn.

2. (a) (i) $(x, y) \to (-x, -y)$; (ii) $(r, \theta) \to (r, \theta + 180°)$.
(b) (i) $(x, y) \to (-y, x)$; (ii) $(r, \theta) \to (r, \theta + 90°)$.

§2.5

(i) $(r, \theta) \to (r, \theta + \alpha)$;
(ii) $(x, y) \to (x \cos \alpha - y \sin \alpha + y \cos \alpha)$.

The transformed curve is $f(r, \theta - \alpha) = 0$.

§2.6

1. If O is the center of a suitable one of the two squares that can be drawn on BC, the first quarter-turn is the product of reflections in CO and CB whereas the second is the product of reflections in BC and BO.

2. P is transformed into A by a quarter-turn about C, and thence into S by a quarter-turn about B.

§2.7

1. (a) C_1, (b) D_1, (c) D_1, (d) C_2,
(e) D_2, (f) D_2, (g) D_2.

2. $\mathrm{RTT}^{-1} = \mathrm{STT}^{-1}$.

3. $\mathrm{R_1R_2 = R_1R_2R_1{}^2 = R_2R_1R_2R_1 = R_2{}^2R_1R_2}$; therefore $\mathrm{R_2{}^2} = 1$ and

$$\mathrm{(R_1R_2)^3 = R_1R_2R_1 \cdot R_2R_1R_2 = (R_1R_2R_1)^2} = 1.$$

4. The periods of the elements of C_n are divisors of n.

§2.8

2. If the angles are all equal, sides of two different lengths can only occur alternately, and this is impossible if their number is odd.

3. $108°, 36°, 140°, 100°, 20°$.

4. Circumradii $l\sqrt{4 \pm 2\sqrt{2}}$, $l\sqrt{2}(\sqrt{3} \pm 1)$; inradii $l(\sqrt{2} \pm 1)$, $l(2 \pm \sqrt{3})$; vertex figures $l\sqrt{2 \pm \sqrt{2}}$, $l(\sqrt{3} \pm 1)/\sqrt{2}$.

5. $\left(1, \dfrac{2k\pi}{n}\right)$.

6. Yes. Make every cut from the center. If the perimeter is divided into equal parts, the area is automatically divided into equal parts.

§3.1

1. Rotation, translation.

2. Reflection. Yes.

3. If the perpendicular bisectors are distinct, find where they intersect. (If they are parallel, the segments are not related by a rotation!) If they coincide, find where the lines AB and $A'B'$ intersect. If these lines also coincide, the center is the midpoint of AA'.

4. Rotate the first two mirrors until the second (in its new position) coincides with the third.

§3.2

1. T^{-1}.

2. It can be any line perpendicular to the direction of the translation.

3. A translation.

4. Translate the first two mirrors till the second (in its new position) coincides with the third.

5. Any translation is expressible as the product of two half-turns, one of which may be arbitrarily assigned. Therefore, if H_1, H_2, H_3 are half-turns,

$$H_2H_3 = H_1H_4$$

for a suitable H_4; that is, $H_1H_2H_3 = H_4$.

7. $(x, y) \to (x + a, y)$. The transformed curve is $f(x - a, y) = 0$; for instance, the unit circle with center $(a, 0)$ is

$$(x - a)^2 + y^2 - 1 = 0.$$

§3.3

1. (i) Half-turn about B, or reflection in the perpendicular line through B.
(ii) Translation from A to B, or a glide reflection.

2. Two is the only even number less than or equal to 3. The product of a reflection and a half-turn is a reflection or a glide reflection according as the center of the half-turn does or does not lie on the mirror.

3. Reflection in the perpendicular line through O.

4. Half-turn.

5. An opposite transformation.

6. The relation $R_1R_2R_3 = R$ is equivalent to $R_1R_2 = RR_3$, which means that R_1R_2 and RR_3 are either equal rotations or equal translations.

7. $G^2 = TR_1R_1T = T^2$.

8. A glide reflection. An equation requires an expression for the *old* coordinates in terms of the *new*.

§3.4

(a) $O_1O_2O_3O_4$ is a parallelogram (possibly collapsing like $OO'Q'Q$ in Figure 3.2*a*).

(b) If m_1 and m_2 intersect, m bisects one of the angles between them. If instead they are parallel, m is parallel to them and midway between them.

§3.5

1. S is a glide reflection.

2. If a translation commutes with a reflection, its direction must be along the mirror.

§3.7

1. (i), (ii), (iii), (iv), (v).

2. (iii), (v).

§4.1

1. Each side of a Dirichlet region joins the circumcenters of two congruent triangles having a common side.

§4.2

1. Because two opposite vertices of each quadrangle are related by a translation.

§4.3

2. Place the two parallelograms in such a position that a side of one is part of a side of the other (with a common vertex at one end).

§4.4

No. A "Procrustean stretch" [Coxeter and Greitzer **1**, p. 102], which doubles vertical distances while halving horizontal distances, can be applied to three of the patterns in Figure 4.4*a*, namely those illustrating the groups **p2, cmm, pmg**.

§4.5

1. Rotation through the same angle about P'.

2. If T is a translation and S is a rotation whose period is greater than 2, then $S^{-1}TS$ is a translation in a new direction.

§4.6

1. Since the vertex figure is regular, two adjacent faces are alike; therefore any two faces are alike. Since the face is regular, two adjacent vertices are surrounded alike; therefore any two vertices are surrounded alike.

3. No. To complete the lattice we would also need the centers of the hexagons.

§4.7

1. If Q were not between P_2 and P_3, we could obtain another pair having a smaller distance than P_1Q.

2. Use induction over n deriving a set of $n - 1$ points by omitting one of two whose join contains no others.

3. A complete quadrangle with its diagonal points.

§5.1

1. $O(\lambda^{-1})$.

2. It divides O_1O_2 in the ratio $(\lambda_2 - 1):(\lambda_1 - 1)\lambda_2$.

3. (a) $(r, \theta) \to (\lambda r, \theta)$,

(b) $(x, y) \to (\lambda x, \lambda y)$.

4. By similar triangles, $OP'/OP = OA'/OA$.

5. By taking O between A and A'.

§5.2

2. If a common tangent TT' meets the line of centers in O, the dilatation $O(OT'/OT)$ transforms the first circle into the second.

3. It divides $O\,O_1$ in the ratio $(\lambda_1 - 1):(1 - \lambda)$.

§5.3

1. 0, -2, -3, -6.

2. The nine-point center is the same for all.

§5.4

Since any invariant point of a transformation is also an invariant point of the inverse transformation, we lose no generality by considering a similarity $ABC \to A'B'C'$ in

which ABC is the larger of the two given similar triangles. (If it were the smaller, we would alter the notation and consider the inverse similarity instead.) If A and A' coincide, we have already found an invariant point. If not, suppose the similarity transforms A' into A'', A'' into A''', and so on. Let μ denote the ratio of magnification, so that $A'B' = \mu AB$, $0 < \mu < 1$. Then $A'A'' = \mu AA'$, $A''A''' = \mu A'A''$, and so on. The circle with center A and radius $(1 - \mu)^{-1}AA'$ is transformed into the circle with center A' and radius $\mu(1 - \mu)^{-1}AA'$. Since

$$1 + \frac{\mu}{1 - \mu} = \frac{1}{1 - \mu},$$

the former circle encloses the latter. Continuing, we obtain an infinite sequence of circles whose radii $\mu^n(1 - \mu)^{-1}AA'$ tend to zero as n tends to infinity. Since these circles are "nested," their centers $A, A', A'', \ldots$ converge to a *point of accumulation* O. Since the similarity transforms $AA'A'' \cdots$ into $A'A''A''', \ldots$, which is essentially the same sequence, the point O is invariant.

§5.5

1. A dilative rotation, possibly reducing to a dilatation or a rotation.

2. Let P be the point of intersection of the corresponding lines AB, $A'B'$. Let the circles $AA'P$, $BB'P$, which have the common point P, meet again in O. The triangles ABO, $A'B'O$ (possibly collapsing into triads of collinear points) are easily seen to be directly similar. Hence, this point O is the invariant point of the direct similarity $AB \to A'B'$.

If AA' and BB' are parallel, O coincides with P (and the two circles have a common tangent at that point). In any other case, an alternative construction makes use of the point T where AA' meets BB'. The circles ABT, $A'B'T$, which have the common point T, meet again in O.

It follows that the four circles $AA'P$, $BB'P$, ABT, $A'B'T$ all pass through one point [Baker 1, p. 110].

§5.6

1. This is the invariant point of a similarity.

2. The segments AB and $A'B'$ are related by a direct similarity and an opposite similarity. The latter is a dilative reflection whose two axes divide each segment PP' in the ratio $AB:A'B'$ (one internally and the other externally; see Figure 5.6*a*). If A_1 coincides with B_1, or A_2 with B_2, the direct similarity is a dilatation, and the analogous points P_1 or P_2 (respectively) all coincide.

3. If S is reflection, S^2 is the identity. If S is a glide reflection, S^2 is a translation, which is one kind of dilatation. If S is a dilative reflection, S^2 is a central diltation.

4. (a) A dilative rotation, possibly reducing to a dilatation or a rotation. (b) A dilative reflection or a glide reflection, possibly reducing to a reflection.

5. As we saw in §5.4, the invariant lines of the dilative reflection meet in the invariant point O and are the internal and external bisectors of $\angle AOA'$. By Euclid VI.3 and its "external" analogue, they are the lines OA_1 and OA_2. The same reasoning can be repeated using the B's instead of the A's.

6. (a) A dilative rotation. (b) A dilative reflection.

§6.1

2. Draw two circles with centers O, A and radius OA, meeting in C, C'. The circle with center C and radius CC' determines B on the circle OCC'.

3. Find the inverse of a point distant $\frac{2}{3}k$ from O and double its distance from O. For a point whose distance from O lies between $k/2n$ and $k/(2n - 1)$, apply the dilatation $O(n)$, invert and then apply $O(n)$ again.

4. To bisect OA, construct B as in ex. 2, and use three more circles to construct its inverse in the circle with center O.

5. To divide OA into n equal parts, transform A by the dilatation $O(n)$, and invert with respect to the circle with center O and radius OA.

§6.3

1. Compare Figure 6.3*b* with Figure 5.2*a*. In the case of equal intersecting circles, one of the inversions is replaced by a reflection.

2. Let Q denote the center of the rhombus $APBP'$. Then

$$\begin{aligned} OP \times OP' &= (OQ - PQ)(OQ + PQ) = OQ^2 - PQ^2 \\ &= OQ^2 + AQ^2 - (AQ^2 + PQ^2) \\ &= OA^2 - PA^2. \end{aligned}$$

3. Let N be the midpoint of BD, and H the foot of the perpendicular from A on the line BD. Suppose $AO = \mu AB$, so that $OP = \mu BD$ and $OP' = (1 - \mu)AC$. Then

$$\begin{aligned} BD \times AC &= (HD - HB)(HD + HB) = HD^2 - HB^2 \\ &= AD^2 - AB^2 \end{aligned}$$

and

$$OP \times OP' = \mu BD \times (1 - \mu)AC = \mu(1 - \mu)(AD^2 - AB^2).$$

4. Let d be the distance from O to the center of γ. Comparing the diameter through O of γ with the corresponding diameter of the inverse circle, we see that the latter is of length

$$\frac{k^2}{d - r} - \frac{k^2}{d + r} = \frac{2k^2r}{d^2 - r^2} = \frac{2k^2r}{p}.$$

§6.5

1. Orthogonal circles invert into orthogonal circles, and any circle orthogonal to the circle of inversion inverts into itself.

2. The two limiting points are the common points of any two members of the orthogonal pencil.

3. Let α_1, α_2 be the two given circles, and β_1, β_2 any two circles that cut them both. Let l_{ij} be the radical axis of α_i and β_j. Let P_j be the point where l_{1j} meets l_{2j}. Then P_1P_2 is the radical axis of α_1 and α_2.

4. One is the center and the other is the point at infinity.

5. Invert the whole figure in a circle whose center is one of the points of contact. Two circles of the ring become parallel lines, say a and b. The rest have their centers and points of contact on a line l, perpendicular to a and b. The original circles become equal circles both touching a and b. The line l serves as a mirror reflecting these two circles into each other. The inversion transforms this reflection into an inversion.

The centers of the circles lie on an ellipse.

6. Inverting in a circle whose center is the point of contact of the tangent circles with centers A and B, we obtain two parallel lines E_1, E_2, and a circle E_3 sandwiched between them (as in the first part of the answer to ex. 9 of §1.5). In this very simple case, Soddy's circles (in the same sandwich) are congruent to E_3 and tangent to it on opposite sides. Their radical axis H_4, joining the points of contact of E_3 with E_1 with E_2, is the inverse of the incircle.

§6.6

1. Each circle of Apollonius is orthogonal to all the circles through A and A'.

2. With its center where the perpendicular bisector of AA' meets l, draw a circle through A (and A'), meeting l in P_1 (near A) and P_2 (near A'). Consider the value of the ratio

$$\mu = \frac{A'P}{AP}$$

for various positions of P. Since P_1P_2 is a diameter of the circle drawn through A and A', two of the circles of Apollonius touch l at P_1 and P_2 respectively. Let μ_1 and μ_2 be the values of μ on these two circles.

On all circles of Apollonius inside the one through P_1, we have $\mu > \mu_1$, and on all circles outside, $\mu < \mu_1$. Therefore, among the various positions of P on l, P_1 has the maximal μ, namely $\mu = \mu_1$. Similarly, the circle of Apollonius through P_2 has $\mu < \mu_2$ inside and $\mu > \mu_2$ outside; therefore, among all positions of P on l, P_2 has the minimal μ, namely $\mu = \mu_2$.

3. $\mu/(1 - \mu^2)$.

4. Since O and $\bar{O}$ are the common points of two circles of Apollonius, we have

$$\frac{OA'}{OA} = \frac{OB'}{OB} = \frac{A'B'}{AB}$$

and the same with O replaced by $\bar{O}$. If $A' = B$, use ex. 3 on p. 76.

5. Inverting in a circle whose center is either of the limiting points of the coaxal pencil, we obtain three *concentric* circles whose radii satisfy either $a_1 < a_2 < a_3$ or $a_1 > a_2 > a_3$. Choosing the limiting point that yields the former order, we find

$$(\alpha_1, \alpha_2) + (\alpha_2, \alpha_3) = \log\frac{a_2}{a_1} + \log\frac{a_3}{a_2} = \log\frac{a_3}{a_1}$$
$$= (\alpha_1, \alpha_3).$$

For further details see Coxeter and Greitzer **1**, pp. 123–131.

6. The circle of similitude is a circle of Apollonius, namely, the locus of a point whose distances from the centers of the two given circles are proportional to their radii, say $OA:OA' = r:r'$. The direct or opposite similarity that transforms OA into OA' also transforms the given circle with center A into the given circle with center A'. It follows that the locus of points from which the two given circles *subtend equal angles* is their circle of similitude or, if the two circles intersect, it is the part of their circle of similitude that lies outside them.

For two *equal* circles, the locus reduces to their radical axis.

7. Let the given circles have centers A, B and radii a, b. Let P, Q be the inverses of a point W on the circle of similitude. Then

$$\frac{AP \times AW}{BQ \times BW} = \frac{a^2}{b^2} = \left(\frac{a}{b}\right)^2 = \left(\frac{AW}{BW}\right)^2,$$

$$\frac{AP}{BQ} = \frac{AW}{BW},$$

and PQ is parallel to AB. Since both the given circles are orthogonal to the circle WPQ, their radical axis is a diameter of the latter, namely the diameter perpendicular

to AB. Since this diameter is also perpendicular to PQ, P and Q are images of each other by reflection in it.

§6.7

$J_k S = S \cdot S^{-1} J_k S = SJ'_k$, where J'_k is the inversion in the circle with center O^S and radius k. The two inversions are the same if and only if the isometry S leaves O invariant, that is, if and only if T interchanges O and O'.

§6.8

1. $OA \times OA' = OB \times OB'$ and $\angle AOB = \angle A'OB'$.

2. The ratio of magnification is

$$\frac{OA'}{OB} = \frac{OA \times OA'}{OA \times OB} = \frac{k^2}{ab}.$$

3. Let $a = OA$, $b = OB$, $c = OC$, $d = OD$. Then

$$\frac{A'B' \times C'D'}{A'C' \times B'D'} = \frac{(k^2/ab)AB \times (k^2/cd)CD}{(k^2/ac)AC \times (k^2/bd)BD} = \frac{AB \times CD}{AC \times BD}.$$

4. Spheres through O invert into planes. Two spheres that touch each other at O have no other common point. Two planes that have no common point are parallel.

5. After inversion we have a sphere γ "sandwiched" between two parallel planes α and β. All the spheres $\sigma_1, \sigma_2, \ldots$ are congruent to γ. The section of the figure by the plane midway between α and β is a circle touching a ring of six congruent circles.

§6.9

1. Consider, for instance, two circles of radius $\frac{1}{3}\pi$ whose centers are distant $\frac{1}{2}\pi$. When each circle is represented by a pair of parallel small circles on a sphere, the points of intersecton are the vertices of 2 squares. (The common radius of the circles may be taken to have any value between $\frac{1}{4}\pi$ and $\frac{1}{2}\pi$. The result is most evident when this value is nearly $\frac{1}{2}\pi$.)

2. Prove this first for a triangle and then dissect the p-gon into triangles.

§7.1

Let R denote the reflection in the plane of the two lines, R_1 and R_2 the reflections in the planes through the respective lines perpendicular to that plane. Then the half-turns may be expressed as R_1R and RR_2, so that their product is R_1R_2.

§7.2

The identity.

§7.3

The reflection in another parallel plane.

§7.4

1. The reflection in another plane through the same line.

2. The two tetrahedra $OABC$ and $OA'B'C'$, being congruent, are related either by a

rotation or by a rotatory-inversion. In the former case any point on the axis of rotation would be, like O, equidistant from A and A', from B and B', and from C and C'.

§7.5

1. (a) Reflection, (b) Quarter-turn, (c) Translation, (d) Twist, (e) Glide reflection, (f) Rotatory inversion.

2. In the notation of Figure 7.5*a*, the half-turns are $R'_1R'_2$, $R'_3R'_4$, and their product is the twist $R'_1R'_3 \cdot R'_2R'_4$.

§7.6

1. It is transformed into

$$(\mu x \cos \alpha - \mu y \sin \alpha,\ \mu x \sin \alpha + \mu y \cos \alpha,\ \mu z).$$

2. Axis $x = y = z$. Angle $2\pi/3$.

3. This is a dilative rotation with angle π and ratio $-\lambda$.

4. Yes. Simply use spheres instead of circles. The same proof is applicable in a space of any number of dimensions.

§7.7

An isometry is the product of four or fewer reflections. Any other similarity is the product of a rotation and a dilatation. If the ratio of magnification happens to be negative, we can use instead a rotatory inversion and a direct dilatation. Since a direct dilatation is the product of inversions in two concentric spheres, this makes, altogether, two or three reflections and two inversions.

Finally, the product of an inversion and an isometry is the product of an inversion and r reflections, $r \leqslant 4$.

§8.1

3. If P_i is (x_i, y_i), M_{ij} is

$$\left(\frac{x_i + x_j}{2},\quad \frac{y_i + y_j}{2}\right)$$

and the midpoint of $M_{12}M_{34}$ is

$$\left(\frac{x_1 + x_2 + x_3 + x_4}{4},\quad \frac{y_1 + y_2 + y_3 + y_4}{4}\right).$$

§8.2

1. $r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_2 - \theta_1)$.

2. (r, θ), where

$$r^2 = \tfrac{1}{4}[r_1^2 + r_2^2 + 2r_1r_2 \cos(\theta_1 + \theta_2)]$$

and

$$\tan \theta = \frac{r_1 \sin \theta_1 + r_2 \sin \theta_2}{r_1 \cos \theta_1 + r_2 \cos \theta_2}.$$

3. $\theta = \alpha$.

4. The respectively parallel lines through the origin are $ax + by = 0$ and $a'x + b'y = 0$, or

$$\frac{y}{x} = -\frac{a}{b} \quad \text{and} \quad \frac{y}{x} = -\frac{a'}{b'}.$$

The condition is derived from 8.22 by writing it in the form

$$\frac{y}{x}\frac{y'}{x'} = -1, \qquad \text{that is,} \qquad \frac{a}{b}\frac{a'}{b'} = -1.$$

5. Replacing x and y by $x\cos\alpha - y\sin\alpha$ and $x\sin\alpha + y\cos\alpha$, where $\alpha = \frac{1}{2}\arctan(-\frac{24}{7}) = \arctan(-\frac{3}{4})$, we obtain

$$4(4x + 3y)^2 + 24(4x + 3y)(-3x + 4y) + 11(-3x + 4y)^2 = 125,$$

which reduces to

$$-x^2 + 4y^2 = 1.$$

§8.3

1. $c(x^2 + y^2) + k^2(ax + by) = 0,$
$c(x^2 + y^2) + 2k^2(gx + fy) + k^4 = 0.$

2. $x^2 + y^2 = k^2.$

3. $(x^2 + y^2)^2 = 2a^2(x^2 - y^2),$
$r^2 = 2a^2\cos 2\theta.$

6. From

$$r\cos\theta + b = 2b\cos t - b\cos 2t = 2b\cos t\,(1 - \cos t) + b$$

and

$$r\sin\theta = 2b\sin t - b\sin 2t = 2b\sin t\,(1 - \cos t)$$

we deduce $r^2 = (r\cos\theta)^2 + (r\sin\theta)^2 = 4b^2(1 - \cos t)^2$ and $\tan\theta = \tan t$. When θ is replaced by $-\theta$, r changes from $2b(1 - \cos\theta)$ to $2b(1 + \cos\theta)$. The sum of these distances is $4b$, for all values of θ.

§8.4

1. A parabola.

2. A hyperbola.

3. The half-turn about the center yields a second focus and a second directrix for any central conic. For the ellipse, the two foci lie between the two directrices; for the hyperbola, the foci lie beyond the directrices. Let O and O' denote the foci and K' the foot of the perpendicular from P to the second directrix. Since

$$OP = \epsilon PK \qquad \text{and} \qquad O'P = \epsilon PK'$$

we have, for the ellipse,

$$OP + O'P = \epsilon(PK + PK') = \epsilon KK',$$

and for the hyperbola,

$$O'P - OP = \epsilon(PK' - PK) = \epsilon KK'.$$

4. $\epsilon = \sqrt{1 \mp b^2/a^2}$, $\sqrt{2}$.

5. Since the circumcenter must be equidistant from A and C, we have

$$x^2 + (\tfrac{2}{3}y)^2 = 1 + (\tfrac{1}{3}y)^2,$$
$$x^2 + \tfrac{1}{3}y^2 = 1.$$

6. By the theory of quadratic equations, F is the product of two linear forms if it is indefinite, and a perfect square if it is semidefinite.

7. $2xy = a^2$.

8. For each point P on the ellipse there is a corresponding point P' on the *auxiliary* circle whose diameter is the major axis, PP' being perpendicular to that axis. The radius through P' makes an angle t with the major axis.

9. Both branches are included.

10. Replace r by l^2/r.

§8.5

1. The parabola $x = 2lt^2$, $y = 2lt$ meets the line $Xx + Yy + Z = 0$ in points given by the roots of the quadratic equation

$$X \cdot 2lt^2 + Y \cdot 2lt + Z = 0.$$

The sum and product of the roots, say t and t', are

$$-\frac{Y}{X} = t + t', \qquad \frac{Z}{2lX} = tt'.$$

2. The secant of the ellipse $x = a\cos t$, $y = b\sin t$ is

$$\begin{vmatrix} x & y & 1 \\ a\cos(\alpha+\beta) & b\sin(\alpha+\beta) & 1 \\ a\cos(\alpha-\beta) & b\sin(\alpha-\beta) & 1 \end{vmatrix} = 0.$$

For the hyperbola, the tangent is

$$\frac{x}{a}\cosh t - \frac{y}{b}\sinh t = 1.$$

3. The envelope of the line $Xx + Yy + Z = 0$, whose coefficients X, Y, Z are functions of a parameter t, is the locus of its point of intersection with

$$(X + dX)x + (Y + dY)y + (Z + dZ) = 0,$$

or with $X'x + Y'y + Z' = 0$, where $X' = \partial X/\partial t$, etc. Differentiating

$$ax\sec t - by\csc t = a^2 - b^2$$

and dividing by $\sin t\cos t$, we obtain

$$\frac{ax}{\cos^3 t} = \frac{-by}{\sin^3 t} = \frac{ax\sec t - by\csc t}{\cos^2 t + \sin^2 t} = a^2 - b^2.$$

Thus the envelope of normals is the locus of (x, y) where

$$\frac{ax}{a^2 - b^2} = \cos^3 t, \qquad \frac{by}{a^2 - b^2} = -\sin^3 t.$$

§8.6

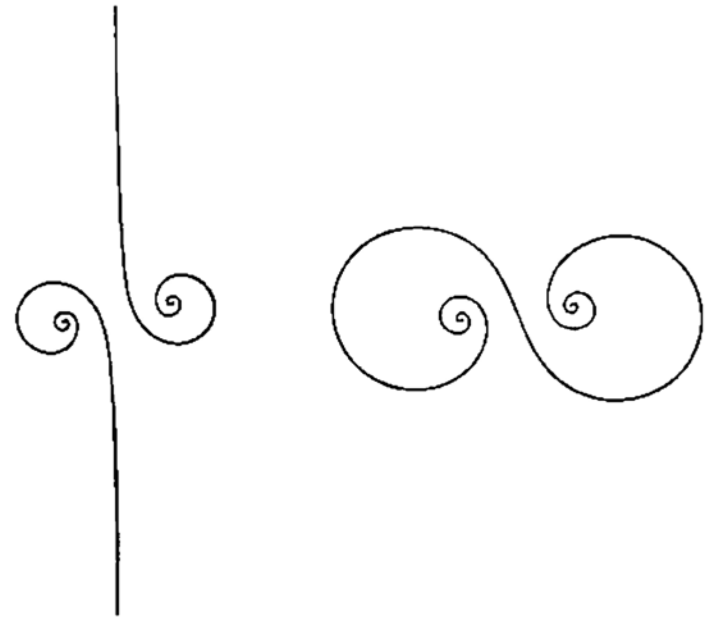
1. πab.

2. $\frac{1}{2}tab$.

§8.7

1. $\mu^{2\pi}r = a\mu^{\theta+2\pi}$. This inversion has the same effect as reflection in the initial line.

2. (i) (ii)



These drawings were made by Ryszard Krasnodębski. See also Coxeter, *Mathematical Gazette*, **52** (1968), p. 5; *Aequationes Mathematicae*, **1** (1968), pp. 112–114.

§8.8

1. Eliminate X: Y:Z:T from the four equations

$$Xx_i + Yy_i + Zz_i = T \qquad \text{and} \qquad Xx + Yy + Zz = T.$$

The effect of replacing the ith point by a direction is the same as the effect of replacing it by $(X_i t,\ Y_i t,\ Z_i t)$, so that the ith row becomes

$$X_i t \quad Y_i t \quad Z_i t \quad 1$$

or (equally well)

$$X_i \quad Y_i \quad Z_i \quad 1/t,$$

and then making $1/t$ tend to zero.

5. The condition for the radii to any common point (x, y, z) of the two spheres to be perpendicular is

$$(x + u)(x + u') + (y + v)(y + v') + (z + w)(z + w') = 0.$$

The desired condition is obtained by doubling this and subtracting the equations of the spheres.

6. The polar plane of (X, Y, Z) passes through (X', Y', Z') if

$$XX' + YY' + ZZ' = k^2.$$

The symmetrical nature of this condition shows that then the polar plane of (X', Y', Z') passes through (X, Y, Z). (Two such points are said to be *conjugate* with respect to the sphere.) The special case asked for arises when (X', Y', Z') lies on the sphere and (X, Y, Z) lies in the tangent plane at that point.

7. Factorizing both sides of the equation

$$\left(\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha\right)^2 - \left(\frac{z}{c}\right)^2 = -\left(\frac{y}{b}\cos\alpha - \frac{x}{a}\sin\alpha\right)^2 + 1,$$

we see that, for each value of α, every point on the line

$$\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = \frac{z}{c}, \qquad \frac{y}{b}\cos\alpha - \frac{x}{a}\sin\alpha = 1$$

lies on the hyperboloid. Reversing the sign of Z, we obtain the other system of generators. The general generator of the first system meets the special generator

$$\frac{x}{a} = -\frac{z}{c}, \qquad \frac{y}{b} = 1$$

of the second where

$$\sin\alpha = \frac{z}{c}(1 + \cos\alpha), \qquad \frac{z}{c}\sin\alpha = 1 - \cos\alpha.$$

The consistency of these equations shows that generators of opposite systems intersect (or else, if $\alpha = \pi$, are parallel). On the other hand, any common point of the general generator of the first system and the special generator

$$\frac{x}{a} = \frac{z}{c}, \qquad \frac{y}{b} = 1$$

of the *same* system would have to satisfy both the equations

$$\sin\alpha = \frac{z}{c}(1 - \cos\alpha), \qquad -\frac{z}{c}\sin\alpha = 1 - \cos\alpha,$$

which can happen only when the two generators coincide.

§9.3

1. $z = 2 \pm i$.

2. $u + vi = 0$ means that the point (u, v) coincides with the origin $(0, 0)$.

3. $(a + bi)^{-1} = x + yi$, where

$$x = \frac{a}{a^2 + b^2}, \qquad y = -\frac{b}{a^2 + b^2}.$$

4. (i) The dilative rotation reduces to a dilatation, and the two shaded triangles are homothetic.

(ii) The dilative rotation reduces to a rotation, and the two shaded triangles are congruent.

§9.4

(a) $3 + 4i = 5(\cos\alpha + i\sin\alpha)$, where $\cos\alpha = \frac{3}{5}$, so that $\cos\frac{1}{2}\alpha = \sqrt{\frac{4}{5}}$ and $\sin\frac{1}{2}\alpha = \sqrt{\frac{1}{5}}$. Hence one square root is

$$(3 + 4i)^{\frac{1}{2}} = 5^{\frac{1}{2}}(\cos\tfrac{1}{2}\alpha + i\sin\tfrac{1}{2}\alpha) = 2 + i$$

and the other is $-(2 + i) = -2 - i$.

(b) $1 = \cos 0 + i \sin 0 = \cos 2\pi + i \sin 2\pi = \cos 4\pi + i \sin 4\pi$. Hence the three cube roots are

$$\begin{aligned} \cos 0 + i \sin 0 &= 1, \\ \cos \tfrac{2}{3}\pi + i \sin \tfrac{2}{3}\pi &= \tfrac{1}{2}(-1 + i\sqrt{3}), \\ \cos \tfrac{4}{3}\pi + i \sin \tfrac{4}{3}\pi &= \tfrac{1}{2}(-1 - i\sqrt{3}). \end{aligned}$$

(c) ± 1, $\pm\omega$, $\pm\omega^2$.

(d) The same and also $\pm i$, $\pm i\omega$, $\pm i\omega^2$.

§9.5

1. $e^{\frac{1}{2}\pi i} = \cos \frac{1}{2}\pi + i \sin \frac{1}{2}\pi = i$. Yes.

§9.6

By Pythagoras,

$$\left(\frac{24}{7}\right)^2 = (x + 1)^2 + \left(1 + \frac{1}{x}\right)^2 = \left(x + 1 + \frac{1}{x}\right)^2 - 1,$$

$$x + 1 + \frac{1}{x} = \frac{25}{7}, \qquad x = \frac{9 + \sqrt{32}}{7}.$$

§9.7

1. The angle is am a

2. The angle is am a.

§10.1

1. The octahedron is a square dipyramid with equilateral side faces.

2. A triangular dipyramid.

3. (i) A square, (ii) a hexagon, (iii) a decagon.

4. The cutting plane, for either acute corner, passes through the midpoints of the three edges that meet there. The rhombohedron, like any parallelepiped, can be repeated by translations to fill the whole space without interstices.

§10.2

1. The bases appear as two pentagons: a large one with a small one oppositely placed inside.

2. Seven.

3. Eight.

§10.3

1. Use 10.32.

2. If a polyhedron has $p \geqslant 4$ for every face and $q \geqslant 4$ for every vertex,

$$\begin{aligned} 4(V - E + F) \leqslant \Sigma q - 4E + \Sigma p &= 2E - 4E + 2E \\ &= 0. \end{aligned}$$

3. The only possibility is a tessellation of rhombi whose vertices form a lattice.

§10.5

1. The edges at a vertex are mutally orthogonal and of equal length.

2. One vertex in each "octant."

3. $(1, 1, 1)$.

4. From the cube in ex. 1 we derive the tetrahedron

$$(0, 0, 0)(0, 1, 1)(1, 0, 1)(1, 1, 0),$$

with face planes $x + y + z = 2$, $x = y + z$, $y = z + x$, $z = x + y$. To normalize these equations we divide by $\sqrt{3}$. The cosine of the internal angle between two of the planes is $\frac{1}{3}$.

5. $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$. The face planes are $\pm x \pm y \pm z = 1$. The edge joining $(1, 0, 0)$ and $(0, 1, 0)$ belongs to the planes $x + y \pm z = 1$, which make an angle whose cosine is $-\frac{1}{3}$.

6. Points between the parallel planes $x + y + z = \pm 1$ satisfy $x + y + z < 1$ and $-x - y - z < 1$; similarly for the other pairs.

7. $120°$. The regular tetrahedron

$$(-1, -1, -1)(-1, 1, 1)(1, -1, 1)(1, 1, -1)$$

has its center at the origin. The planes joining the origin to pairs of vertices are

$$y \pm z = 0, \qquad z \pm x = 0, \qquad x \pm y = 0.$$

§11.1

1. $\tau \sin \dfrac{\pi}{5} = \sin \dfrac{2\pi}{5} = 2 \cos \dfrac{\pi}{5} \sin \dfrac{\pi}{5}$.

2. Center S, radius QU.

§11.2

1. The four points $(\pm\tau, \pm 1, 0)$ evidently form a golden rectangle in the plane $z = 0$.

2. The segment $(0, 0, \tau^2)(0, \tau^2, 0)$ is divided in the ratio $\tau:1$ by the point $(0, \tau, 1)$.

3. $(0, \pm\tau^{-1}, \pm\tau)$, $(\pm\tau, 0, \pm\tau^{-1})$, $(\pm\tau^{-1}, \pm\tau, 0)$, $(\pm 1, \pm 1, \pm 1)$. Thus the 20 vertices belong to 3 "doubly golden" rectangles (whose sides are in the ratio $\tau^2:1$) and a cube.

§11.3

1. Corresponding sides of the two rectangles meet in the points B, D, F, H. The lines BF and DH meet in O.

2. The points I, G, C, A, having polar coordinates $(\tau^{-2}, -\pi)$, $(\tau^{-1}, -\frac{1}{2}\pi)$, $(\tau, \frac{1}{2}\pi)$, (τ^2, π), have Cartesian coordinates $(-\tau^{-2}, 0)$, $(0, -\tau^{-1})$, $(0, \tau)$, $(-\tau^2, 0)$. Hence the lines IC, GA are

$$\tau^3 x - y + \tau = 0, \qquad x + \tau^3 y + \tau^2 = 0;$$

and H, their point of intersection, is $(-\frac{1}{2}, -\frac{1}{2})$ or, in polar coordinates, $(\sqrt{\frac{1}{2}}, -\frac{3}{4}\pi)$. Thus the points J, H, F, D, B, given by

$$r = 2^{-\frac{1}{2}}\tau^n, \qquad \theta = \tfrac{1}{4}(2n - 3)\pi \qquad (n = -1, 0, 1, 2, 3),$$

lie on the spiral 8.71, where $a = 2^{-\frac{1}{2}}\tau^{\frac{3}{2}}$. This is derived from the spiral $r = \mu^\theta$ by the dilatation $O(a)$ or by rotation through the angle

$$\frac{\log a}{\log \mu}.$$

§11.4

1. $f_{n+2} - 1$.

2. Using induction, assume $f_{n-2}f_n - f_{n-1}^2 = (-1)^{n-1}$. Then

$$f_{n-1}f_{n+1} - f_n^2 = f_{n-1}(f_{n-1} + f_n) - f_n(f_{n-2} + f_{n-1})$$
$$= f_{n-1}^2 - f_{n-2}f_n = (-1)^n.$$

3. Working modulo 10, we have $5 + 8 \equiv 3$, $8 + 3 \equiv 1$, and so on.

4. Setting $k + j = n$, we see that the coefficient of t^n is

$$\Sigma\binom{k}{j}$$

where $k = n - j$ and, since $2j \leqslant j + k = n$, $0 \leqslant j \leqslant \frac{1}{2}n$.

5. 1.010203050813213455 . . . .

6. By 11.48,

$$\frac{g_{n+1}}{g_n} = \frac{\tau^{n-1} + (-\tau)^{-n-1}}{\tau^n + (-\tau)^{-n}} = \frac{\tau + (-1)^{n-1}\tau^{-2n-1}}{1 + (-1)^n\tau^{-2n}}.$$

Hence the limit is τ.

§11.5

When $k = 1$, 11.51 yields $h = \tau^{-3/2} = 0.48587 \ldots$.

§12.1

1. Both.

2. The sum of the three angles of a triangle is equal to two right angles.

3. Affine geometry.

4. (a) Affine, (b) absolute, (c) absolute.

§12.2

1. By Axiom 12.22, there is a point C with $[ABC]$, also a point D with $[BCD]$, and so on forever.

2. Theorem 12.271.

3. If $[FDE]$, we could apply 12.27 to the triangle BFD with $[DCB]$, obtaining Z on EC with $[BZF]$. Since $Z = A$, this contradicts $[AFB]$.

4. Any line not belonging to the set contains an infinite number of points, among which only a finite number can lie on lines of the set (at most one on each).

5. Use 12.278 (and Fig. 12.2*d*) with D, A, B, F, C, L replaced by A, B, C, L, M, N.

6. Take A' on A/B, B' on B/C, and apply Axiom 12.27 to the triangle $A'B'B$ with $[B'BC]$ and $[BAA']$.

7. For *any* such A' and B', the line $A'B'$ meets A/C; therefore it does not meet C/A.

§12.3

The $n - 1$ points P_i $(i > 1)$ are joined in pairs by at most $\binom{n-1}{2}$ lines, some or all of which may meet P_1Q. In Figure 12.3*b*, the six joins P_3P_4, P_4P_1, P_1P_2, P_3P_6, P_6P_1, P_1P_3 all make the same contribution as P_1P_5.

§12.4

1. Since the five points are not collinear, they must form either a convex pentagon or a convex quadrangle with one point inside, or a triangle with two points inside. In

the first and second cases the result is obvious. In the third case the two inner points lie on a line meeting two distinct sides of the triangle. The ends of the third side form, with these inner points, the desired convex quadrangle.

4. The first two lines, BC and CA, decompose the plane into four angular regions. The line AB has no intersection with the region bounded by C/A and C/B, but it decomposes each of the remaining three angular regions into two parts. The region bounded by the triangle is the only finite part, since at least one side of each of the others is a ray.

5. Consider any $m - 1$ of the m lines, and the $f(2, m - 1)$ regions formed by them. They decompose the mth line into m parts (namely, two rays and $m - 2$ segments), lying respectively in m of the $f(2, m - 1)$ regions. These m regions are each decomposed into two, whereas the rest are unaffected. Hence

$$f(2, m) = f(2, m - 1) + m.$$

Combining this with the analogous equations

$$\begin{aligned} f(2, m - 1) &= f(2, m - 2) + m - 1, \\ \cdots \quad &\quad \cdots \\ f(2, 1) &= f(2, 0) + 1, \end{aligned}$$

we obtain

$$\begin{aligned} f(2, m) &= f(2, m - 1) + m \\ &= f(2, m - 2) + (m - 1) + m \\ &= \cdots \\ &= f(2, 0) + 1 + 2 + \cdots + m \\ &= 1 + \binom{m + 1}{2} = \binom{m}{0} + \binom{m}{1} + \binom{m}{2}. \end{aligned}$$

6. The first $m - 1$ planes decompose the mth into $f(2, m - 1)$ plane regions lying respectively in $f(2, m - 1)$ of the $f(3, m - 1)$ solid regions. These $f(2, m - 1)$ solid regions are each decomposed into two, whereas the rest are unaffected. Hence

$$\begin{aligned} f(3, m) &= f(3, m - 1) + f(2, m - 1) \\ &= f(3, m - 2) + f(2, m - 2) + f(2, m - 1) \\ &= \cdots \\ &= f(3, 0) + f(2, 0) + f(2, 1) + \cdots + f(2, m - 1) \\ &= 1 + \sum_{r=1}^{m} \left\{ 1 + \binom{r}{2} \right\} = 1 + m + \binom{m + 1}{3} \\ &= \binom{m + 1}{1} + \binom{m + 1}{3} \\ &= \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \binom{m}{3}. \end{aligned}$$

7. $f(n, m) = f(n, m - 1) + f(n - 1, m - 1)$, with $f(n, 0) = 1$. Therefore

$$f(n, m) = f(n, 0) + \sum_{r=0}^{m-1} f(n - 1, r).$$

To prove

$$f(n, m) = \binom{m}{0} + \binom{m}{1} + \cdots + \binom{m}{n}$$

by induction, assume the same formula with n replaced by $n-1$.

$$\begin{aligned} f(n, m) &= f(n, 0) + \sum_{r=0}^{m-1}\left\{\binom{r}{0} + \binom{r}{1} + \cdots + \binom{r}{n-1}\right\} \\ &= 1 + \sum_{0}^{m-1}\binom{r}{0} + \sum_{1}^{m-1}\binom{r}{1} + \cdots + \sum_{n-1}^{m-1}\binom{r}{n-1} \\ &= \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \cdots + \binom{m}{n}. \end{aligned}$$

The final step makes use of the familiar series

$$\sum_{r=n-1}^{m-1}\binom{r}{n-1} = \sum_{n-1}^{m-1}\left\{\binom{r+1}{n} - \binom{r}{n}\right\} = \binom{m}{n} - \binom{n-1}{n} = \binom{m}{n}.$$

§12.6

1. The relation $[prs]$ tells us that the three lines do not meet one another and that they contain points A, C, B (respectively) such that $[ACB]$. Suppose that p_1 is parallel to s. Then any ray from A within the angle between AB and p_1 meets s, and therefore also r. Hence p_1 is parallel to r.

2. The two rays through a given point parallel to a given line appear as the segments joining the point to the ends of the chord.

§13.1

1. Let q and r be the two parallel lines. If p met q without meeting r, then p and q would be two lines through the point $p \cdot q$, both parallel to r, contradicting Axiom 13.11.

2. Yes, provided the three lines are coplanar.

§13.2

1. This is a one-dimensional version of the principle that every direct isometry (including the identity) is the product of an even number of opposite isometries. Translations preserve directions, whereas half-turns reverse directions.

2. By 13.25, $(A \rightarrow D) = (C \rightarrow B)$ implies $(A \leftrightarrow B) = (C \leftrightarrow D)$, that is, $(A \leftrightarrow B) = (D \leftrightarrow C)$, which, in turn, implies $(A \rightarrow C) = (D \rightarrow B)$.

3. This is the half-turn about C.

4. Any two opposite sides are interchanged by the half-turn about the common midpoint of the diagonals.

5. Using the symbol $\equiv$ to relate congruent segments, we have

$$BA_1 \equiv C_1B_1 \equiv A_2C, \qquad BA_2 \equiv C_2B_2 \equiv A_3C.$$

By ex. 2, $BA_1 \equiv A_2C$ implies $BA_2 \equiv A_1C$. Hence $A_3 = A_1$.

6. Dissect the quadrangle into two triangles by a diagonal, and use 13.26.

7. The three bimedians all have the same midpoint.

§13.3

1. Each point on AB is transformed into a point dividing the segment AB in the same ratio, that is, into itself. For any other point P, we can draw PM parallel to CA, and PN parallel to CB, with M and N on AB. The corresponding point P' is obtained by drawing MP' parallel to AC', and NP' parallel to BC'.

This is a typical *affine construction.* Instead of Euclid's "ruler and compasses" we are using a simpler instrument, the *parallel-ruler,* which consists of four rulers (or two rulers and two auxiliary bars) forming a parallelogram with pivots at the four vertices. This enables us to draw lines parallel to a given line (and, of course, also to join two given points).

2. If an affinity interchanges two points C and C', it leaves invariant the midpoint O of CC'. If O is the only invariant point, it is also the midpoint of PP' for any P, and the affinity is the half-turn $C \leftrightarrow C'$. But if there is another invariant point M, the affinity, transforming the triangle MCC' into $MC'C$, is the reflection $M(CC')$.

3. If the affinity is not a dilatation, it must transform at least one line a into a line a' not parallel to a. Another line b, parallel to a, will be transformed into another line b', parallel to a'. The point of intersection $A = a \cdot a'$ is invariant; for, if not, it would lie on an invariant line m and could be called either $a \cdot m$ or $a' \cdot m$, contradicting its noninvariance. Similarly, there is another invariant point $B = b \cdot b'$. Therefore, the affinity is either a shear or a strain: $ABC \to ABC'$.

4. Since lines have linear equations, any transformation of the form 13.33 preserves collinearity. Conversely, we can express any affinity $IXY \to I'X'Y'$ in the form 13.33 with suitable values for a, b, c, d, l, m. For, the triangle $(0, 0)(1, 0)(0, 1)$ is transformed into

$$(l,\ m)(a + l,\ c + m)(b + l,\ d + m),$$

which can be identified with any given triangle. The non-collinearity of these three points is ensured by the condition $ad \neq bc$.

5. (i) Translation, (ii) Central dilatation,
(iii) Shear, (iv) Strain

(including an affine reflection as the special case when $a = -1$).

§13.4

1. This follows from the remark after 13.41.

2. If the diagonals P_0P_3, P_1P_4, P_2P_0, P_3P_1 are parallel to the sides P_1P_2, P_2P_3, P_3P_4 P_4P_0, respectively, the following triangles all have the same area:

$$P_0P_1P_2,\ P_1P_2P_3,\ P_2P_3P_4,\ P_3P_4P_0,\ P_4P_0P_1.$$

Therefore, P_2P_4 is parallel to P_0P_1.

3. When it is a translation or a half-turn. In fact, a central dilatation $O(\lambda)$ has $ad - bc = \lambda^2$.

4. Always.

5. Never.

6. Each affine reflection reverses area.

7. A translation is the product of two affine reflections in the direction of the translation, the mirrors being parallel lines in any other direction. More precisely, in the notation of Figure 13.2*d*, the translation $A \to D$ is the product of reflections $A(BC)$ and $B(AD)$.

A half-turn is the product of reflections in two intersecting lines, the direction of each reflection being along the mirror for the other: the half-turn $A \leftrightarrow B$ is the product of $A(CD)$ and $C(AB)$.

A shear is the product of reflections in one mirror in two different directions. Alternatively, it is the product of reflections in one direction in two intersecting mirrors.

8. For any geometric transformation, the successive transforms of a noninvariant point P_0 comprise a set of points $P_0P_1P_2 \cdots$ called the *orbit* of P_0; the transformation takes P_0 to P_1, P_1 to P_2, and so on. Exercise 3 (on page 203), describing a situation in which the orbit of every point consists of a set of collinear points, shows that the only affinities of this kind are the "trivial" ones: the dilatations, shears and strains. For every other kind of affinity there is at least one noninvariant point P_0 lying on no invariant line; the line P_0P_1 is transformed into a different line P_1P_2, the orbit begins with three points forming a triangle, and the affinity can be expressed as $P_0P_1P_2 \rightarrow P_1P_2P_3$. In the case of an equiaffinity, the "trivial" kinds are those considered in ex. 7. For any other kind, $P_0P_1P_2$ and $P_1P_2P_3$ are two triangles of equal area. Since they have a common side P_1P_2, ex. 1 shows that P_0P_3 must be parallel to P_1P_2.

9. Since the translation, half-turn and shear have already been covered in ex. 7, we may restrict consideration to $P_0P_1P_2 \rightarrow P_1P_2P_3$ with P_0P_3 parallel to P_1P_2. Letting M denote the midpoint of P_0P_3, as in Figure 13.4*b*, we see that this equiaffinity is the product of the two affine reflections

$$\mathrm{R}_1 = M(P_1P_2) \qquad \text{and} \qquad \mathrm{R}_2 = P_2(P_1P_3)$$

(compare §2.7 on page 34). For, these reflections have the effect

$$P_0P_1P_2 \rightarrow P_3P_2P_1 \rightarrow P_1P_2P_3.$$

10. Since R_1 transforms the points $\cdots P_0P_1P_2P_3 \cdots$ into $\cdots P_3P_2P_1P_0 \cdots$, while R_2 transforms $\cdots P_0P_1P_2P_3P_4 \cdots$ into $\cdots P_4P_3P_2P_1P_0 \cdots$, the stated parallelism certainly occurs when $i + j = h + k = 3$ or 4. For other values, we simply have to transform by a suitable power of the equiaffinity $\mathrm{R}_1\mathrm{R}_2$.

11. Because affine geometry cannot distinguish between a circle and any other ellipse. The elliptic shadow cast by a coin illustrates the fact that an ellipse is a "strained circle." We are free to use a strain as a coordinate transformation, writing ϵx for x, so that the ellipse becomes

$$\epsilon^2 x^2 + y^2 = 1$$

and the elliptic rotation 13.49 becomes

$$x' = x\cos\theta - \frac{y}{\epsilon}\sin\theta, \qquad y' = \epsilon x\sin\theta + y\cos\theta.$$

This equiaffinity reduces to a half-turn when we set $\theta = \pi$. If instead we set $\theta = \pi - \epsilon$, $\epsilon = \pi/(2d + 1)$, and make d tend to infinity, we obtain a new equiaffinity: *the focal rotation*

$$x' = -x - y, \qquad y' = -y,$$

which leaves invariant the pair of parallel lines $y^2 = 1$. In other words, the affinely regular star polygons of type $\{2 + 1/d\}$ $(d = 2, 3, 4, \ldots)$ may be regarded as approximating either a digon (page 37) or a *focal polygon*, whose vertices

$$(0, 1), (-1, -1), (2, 1), (-3, -1), \ldots$$

lie alternately on these two parallel lines while its sides pass alternately through the two "foci" $(\mp\frac{1}{2}, 0)$.

12. Every triangle is affinely regular, but the only affinely regular quadrangles are the parallelograms.

13. Given P_0, P_1, P_2, complete the parallelogram $P_0P_1P_2O$. Then draw P_2P_3 parallel

to P_1O (with P_3 on O/P_0), P_3P_4 parallel to P_2O (with P_4 on O/P_1), and P_4P_5 parallel to OP_0 (with P_5 on O/P_2).

14. It is important to remember that affine geometry admits no measure of angles. The symbol θ occurring in 13.49 must not be interpreted as an angle but simply as a number. The use of sines and cosines does not force us to work in Euclidean geometry: they are employed because of their convenient properties, such as $\cos^2 \theta + \sin^2 \theta = 1$. These functions can, of course, be defined by analytical means without any reference to geometry. After these words of caution, we take the typical vertex P_j to have affine coordinates

$$(\cos j\theta,\ \sin j\theta),$$

where $\theta = 2\pi/n$, and conclude that

$$\frac{P_0P_3}{P_1P_2} = \frac{\sin 3\theta}{\sin 2\theta - \sin \theta} = \frac{3 - 4\sin^2 \theta}{2\cos \theta - 1} = 2\cos \theta + 1.$$

15. The values are 2, 3, 4, 6 (as in §4.5 on page 60). This conclusion can be justified as follows. We see from §13.3 that the parallel-ruler enables us to multiply the length of a given segment by any *rational* number. In fact, given $(0, 0)$, $(1, 0)$, $(0, 1)$, the points that can be constructed by means of the parallel-ruler are the points (x, y) whose affine coordinates are rational, and no others. We see from ex. 8 that the nature of an affinely regular polygon $P_0P_1P_2 \cdots$ is determined by the position of P_3 on the line through P_0 parallel to P_1P_2. It is clear from ex. 10 that we can then construct P_4 and all the other vertices in turn. Exercise 14 shows that, for a polygon of type $\{n\}$, P_3 can be constructed if and only if $\cos \theta$ *and* θ/π *are both rational.*

The following trick for determining the admissible values of θ was devised by the same H. W. Richmond who geometrized Gauss's solution of the cyclotomic equation $z^{17} - 1 = 0$ (Figure 2.1*b* on page 27).

Since $\cos 2\theta = 2\cos^2 \theta - 1$, every rational $\cos \theta$ yields a rational $\cos 2\theta$. Since θ/π is rational, the expressions

$$\cos \theta,\ \cos 2\theta,\ \cos 4\theta, \ldots,\ \cos 2^k\theta, \ldots$$

comprise a *finite* set of rational numbers. When these rational numbers are expressed as fractions in their "lowest terms," let b be the greatest denominator that occurs, and let

$$\cos \phi = \frac{a}{b} \qquad (\phi = 2^k\theta)$$

be one of the numbers having this denominator. Since a and b are relatively prime, the denominator of

$$\cos 2\phi = \frac{2a^2 - b^2}{b^2}$$

is either b^2 or (if b is even) $\frac{1}{2}b^2$. But this denominator must not be greater than b. Therefore,

$$b \geqslant \tfrac{1}{2}b^2 > 0,\ b \leqslant 2,\ b = 1 \text{ or } 2,\ \cos \phi = 0 \text{ or } \pm 1 \text{ or } \pm\tfrac{1}{2},$$

and the only admissible values for ϕ are $j\pi/2$ and $j\pi/3$ for integers j. Since $\cos(\pi/4)$ and $\cos(\pi/6)$ are irrational, it follows that the only admissible values for θ, with $0 < \theta \leqslant \pi$, are

$$\pi,\ \frac{2\pi}{3},\ \frac{\pi}{2},\ \frac{\pi}{3}.$$

In our geometric application of this result, $\theta = 2\pi/n \leqslant \pi$. Hence $n = 2, 3, 4$ or 6. In other words, the only finite affinely regular polygons constructible with the parallel-ruler are the digon, triangle, parallelogram, and affinely regular hexagon. (For instance, the pentagram and pentagon shown in Figure 13.4*e* are *not* constructible. It is impossible to assign rational coordinates to all the five vertices simultaneously.)

The connection with §4.5 may be explained by observing that, in the formation of a crystal, Nature is, in effect, using a parallel-ruler to line up certain atoms in the straight rows of a lattice.

§13.5

1. Any common factor of x and x_1 would divide $xy_1 - yx_1$. By 13.52, this is impossible.

2. By 13.52, $x_0 y - y_0 x = 1 = xy_1 - yx_1$, and therefore

$$(x_0 + x_1)y = (y_0 + y_1)x.$$

3. We can systematically assign the symbols 0, 1, . . . , 6 in cyclic order to the points of the basic lattice, as follows: each of A, B, C gets the label 0, and then we proceed with 1, 2, 3, 4, 5, 6, 0, 1, . . . to the right; from A toward L we have 0, 3, 6, 2, 5, 1, 4, 0, . . . ; and from B toward M we have 0, 5, 3, 1, 6, 4, 2, 0, ... All the points numbered alike form a "sublattice," and since there are seven such sublattices, each has a unit cell seven times as big as that of the basic lattice.

Alternatively, let the basic lattice consist of the points whose affine coordinates are integers. Take B at $(0, 0)$, C at $(2, 1)$, A at $(-1, 3)$. Then the only lattice points inside the triangle ABC are $(1, 1)$, $(0, 2)$, $(0, 1)$, forming a triangle of area $\frac{1}{2}$ (that is, half a unit cell). By Pick's theorem, the area of ABC is $\frac{3}{2} + 3 - 1 = \frac{7}{2}$.

3. (a) The triangle $(0, 0)$ $(3, 1)$ $(-1, 4)$ has area $\frac{3}{2} + 6 - 1 = \frac{13}{2}$, whereas the Cevians form an inner triangle of area $\frac{6}{2} + 0 - 1 = 2$. The ratio is $\frac{4}{13}$. (b) The triangle $(0, 0)$ $(3, 2)$ $(-2, 5)$ has area $\frac{3}{2} + 9 - 1 = \frac{19}{2}$, whereas the Cevians form an inner triangle of area $\frac{1}{2}$.

5. The parallelogram $(0, 0)$ $(2, -1)$ $(3, 1)$ $(1, 2)$ has area $\frac{4}{2} + 4 - 1 = 5$, whereas the small parallelogram in the middle has area 1.

The parallelogram $(\pm 6, \pm 6)$ has area 144, whereas the small octagon in the middle, namely $(\pm 3, 0)$ $(\pm 2, \pm 2)$ $(0, \pm 3)$, with 21 interior points, has area 24.

6.
$$\begin{aligned}\frac{1}{\lambda+1}\frac{\mu}{\mu+1} + \frac{1}{\mu+1}\frac{\nu}{\nu+1} + \frac{1}{\nu+1}\frac{\lambda}{\lambda+1} &= \frac{\mu(\nu+1) + \nu(\lambda+1) + \lambda(\mu+1)}{(\lambda+1)(\mu+1)(\nu+1)} \\ &= 1 - \frac{\lambda\mu\nu + 1}{(\lambda+1)(\mu+1)(\nu+1)}.\end{aligned}$$

7. This is obvious unless λ, μ, ν are either all $\geqslant 1$ or all $\leqslant 1$. Assuming one of these eventualities, suppose, if possible, that LMN is the smallest of the four triangles. Then $\lambda\mu\nu + 1$ must be less than or equal to each of

$$(\lambda + 1)\nu, \qquad (\mu + 1)\lambda, \qquad (\nu + 1)\mu.$$

By addition,

$$\begin{aligned}3(\lambda\mu\nu + 1) &\leqslant (\lambda + 1)\nu + (\nu + 1)\lambda + (\nu + 1)\mu \\ &= (\mu\nu + \lambda) + (\nu\lambda + \mu) + (\lambda\mu + \nu),\end{aligned}$$

that is,

$$(\lambda - 1)(\mu\nu - 1) + (\mu - 1)(\nu\lambda - 1) + (\nu - 1)(\lambda\mu - 1) \leqslant 0.$$

Since λ, μ, ν are all $\geqslant 1$ or all $\leqslant 1$, each of these three terms must be zero, so that

$$\lambda = \mu = \nu = 1.$$

§13.6

1. (i), (ii), and (iv) lack the additive identity (zero); (iii) contains 1 and i but not $1 + i$. The remaining four sets include zero, which has no multiplicative inverse.

2. If B is the centroid of masses a at A and c at C, B' is the centroid of masses a at A' and c at C'. Points dividing AA', BB', CC' in the ratio $\mu:1$ are the centroids of masses 1 at A and μ at A', 1 at B and μ at B', 1 at C and μ at C'. Of these three points, the middle one is the centroid of masses a at the first and c at the last.

In the concise notation of Möbius' *barycentric calculus*, we have

$$B = aA + cC, \qquad B' = aA' + cC',$$
$$B + \mu B' = a(A + \mu A') + c(C + \mu C').$$

3. In Möbius' notation, the centroid of equal masses at the four vertices of the quadrangle $ABCD$ is $A + B + C + D$. The vertices of the Varignon parallelogram are $A + B$, $B + C$, $C + D$, $D + A$, and its center is $(A + B) + (C + D) = (B + C) + (D + A)$.

4. Cutting the quadrangle along either diagonal, we obtain two triangles whose centroids are the midpoints of two of the broken line segments in Figure 13.6*b*.

5. A centrally symmetrical quadrangle, that is, a parallelogram.

§13.7

1. Inside the triangle $A_1A_2A_3$ we have $+\ +\ +$; beyond the side A_2A_3, $-\ +\ +$; and beyond the vertex A_1, $+\ -\ -$.

2. $\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ x & y & 1 \end{vmatrix}.$

3. $\tfrac{1}{2}(\overrightarrow{OS} + \overrightarrow{OT}) = \tfrac{1}{2}(\Sigma s_i\, \overrightarrow{OA}_i + \Sigma t_i\, \overrightarrow{OA}_i)$

$$= \Sigma\tfrac{1}{2}(s_i + t_i)\overrightarrow{OA}_i.$$

4. $\sigma\Sigma s_i\overrightarrow{OA}_i + \tau\Sigma t_i\overrightarrow{OA}_i = \Sigma(\sigma s_i + \tau t_i)\overrightarrow{OA}_i.$

5. In this formulation it is no longer necessary to assume $\Sigma s_i = \Sigma t_i$.

6. $\begin{vmatrix} 0 & 1 & \lambda \\ \mu & 0 & 1 \\ 1 & \nu & 0 \end{vmatrix} = \lambda\mu\nu + 1.$

This has to be divided by $(\lambda + 1)(\mu + 1)(\nu + 1)$. When L, M, N are collinear, it becomes zero, in agreement with Menelaus's theorem.

7. When the line is entirely outside the triangle, the signs are all alike (say all plus). When the line penetrates the sides A_2A_3 and A_3A_1, T_3 differs in sign from T_1 and T_2.

§13.8

1. Any common point of a and b would be a common point of a and α. Apply 13.82 to b, c, and a.

2. Since a is parallel to b, it is parallel to the plane α through b, and we can use ex. 1.

3. Our proof of 13.81 shows that all the lines through A in the plane $q'r'$ are parallel to lines in the plane qr.

4. The centroid of equal masses at A_1, A_2, A_3, A_4 is the centroid of masses 1 at A_1, 1 at A_2, and 2 at the midpoint of A_3A_4.

5. (t_1, t_2, t_3, t_4) is the centroid of masses t_i at A_i $(i = 1, 2, 3, 4)$.

§13.9

1. In the notation of Figure 13.8b, if A', B', C', O' are the vertices opposite to A, B, C, O, the six sides of the skew hexagon $AB'CA'BC'$ can be joined to the diagonal OO' to form a cycle of six tetrahedra, consecutive pairs of which are related by affine reflections; e.g., the tetrahedra $OO'AB'$ and $OO'B'C$ are related by the affine reflection $OO'(AC)$, which leaves invariant the plane $OO'B'$ while interchanging A and C.

2. An affine reflection interchanges pairs of points, P and P', in such a way that all the joining lines PP' are parallel and all the segments PP' are bisected by the mirror (which, in the three-dimensional case, is a plane).

3. The points (a, b, c) and (a', b', c') are interchanged by the central inversion $(x, y, z) \to (a + a' - x, b + b' - y, c + c' - z)$.

4. If (x, y, z) is a lattice point lying in a first rational plane $Xx + Yy + Zz = \pm 1$, any common divisor of x, y, z would have to divide ± 1.

5. No. For instance, $(1, 1, 0)$ is a visible point in the "second" rational plane $x + y = 2$.

6. When $x = 1$, we have $2y + 3z = -1$. Two obvious solutions are $y = 1, z = -1$, and $y = -2, z = 1$. When $x = -4$, we have $2y + 3z = 5$, with the obvious solution $y = z = 1$. We thus obtain the triangle $(1, 1, -1)(1, -2, 1)(-4, 1, 1)$.

7. The triangle $(1, 1, -1)(1, -2, 1)(-4, 1, 1)$, whose determinant is -1, is half a unit cell for the lattice in the plane $6x + 10y + 15z = 1$. Hence the general lattice point in this plane is

$$(1, 1, -1) + m(0, -3, 2) + n(-5, 0, 2) = (1 - 5n, 1 - 3m, -1 + 2m + 2n),$$

where m and n run through all the integers.

8. The given equation implies $x^2 + 2\sqrt{2}xy + 2y^2 = 3z^2$. Since $\sqrt{2}$ is irrational, any solution in integers woud require $xy = 0$ and $x^2 + 2y^2 = 3z^2$, which is impossible by the usual argument for establishing the irrationality of $\sqrt{3}$.

§14.1

1. By 14.11 the four points described in 14.13 are joined in pairs by six lines which, by 14.12, meet any other line in at least three points. Also each of the six lines meets the others in three points.

2. The m points, with c lines through each, apparently make a total of cm lines; but in this estimate each of the n lines is counted d times, once for each of the d points on it. Therefore $cm = dn$. The Pappus configuration 9_3 may be regarded as a cycle of three "Graves triangles" in six ways. (Coxeter, *Projective Geometry*, 1964, p. 39.)

3. Here is the table:

12	11	10	9	8	7	6	5	4	3	2	1	0
1	2	3	4	5	6	7	8	9	10	11	12	0
2	3	4	5	6	7	8	9	10	11	12	0	1
4	5	6	7	8	9	10	11	12	0	1	2	3
10	11	12	0	1	2	3	4	5	6	7	8	9

The last column indicates that the points on p_0 are P_0, P_1, P_3, P_9, and that the lines through P_0 are p_0, p_1, p_3, p_9. The other columns have an analogous interpretation. The columns of the table

$$\begin{array}{cccccccc} P_0 & P_{10} & P_9 & P_6 & P_5 & P_2 & P_8 & P_3 \\ P_{10} & P_9 & P_6 & P_5 & P_2 & P_8 & P_3 & P_0 \\ P_6 & P_5 & P_2 & P_8 & P_3 & P_0 & P_{10} & P_9 \end{array}$$

indicate the eight lines of the configuration 8_3 formed by the cycle of eight points $P_0P_{10}P_9P_6P_5P_2P_8P_3$. The two mutually inscribed quadrangles are obtained by taking alternate points of this cycle.

4. Through any one of the points we have $p + 1$ lines, each containing p further points. This makes $1 + (p + 1)p$ points altogether.

5. The whole finite geometry provides a counterexample to refute Sylvester's theorem. Every line joining two of the points belongs to the geometry and thus contains not only two but $p + 1$ of the points.

§14.2

1. Let A, B, C, D be the points 14.23, and continue as follows:

$$\begin{array}{ll} E = AD \cdot BC = (0, 1, 1), & F = BD \cdot CA = (1, 0, 1), \\ G = AB \cdot EF = (-1, 1, 0), & H = BC \cdot DG = (0, 2, 1), \\ I = AD \cdot FH = (1, 2, 2), & J = EF \cdot CI = (1, 2, 3). \end{array}$$

2. The three pairs of opposite sides are

$$x_2 \pm x_3 = 0, \qquad x_3 \pm x_1 = 0, \qquad x_1 \pm x_2 = 0.$$

3. We see that $P_5 = P_2P_3 \cdot P_4P_7$, $P_6 = P_1P_7 \cdot P_3P_4$. The collinearity of $P_0P_5P_6$ makes $x^2 + x + 1 = 0$.

4.
$$\begin{array}{ll} P_4 = P_0P_5 \cdot P_1P_2 = (0, 1, 1), & P_8 = P_1P_5 \cdot P_2P_0 = (1, 0, 1), \\ P_3 = P_2P_5 \cdot P_0P_1 = (1, 1, 0), & P_6 = P_1P_5 \cdot P_3P_4 = (1, 2, 1) \\ P_7 = P_0P_5 \cdot P_3P_8 = (2, 1, 1), & P_9 = P_0P_1 \cdot P_4P_8 = (1, 2, 0), \\ P_{10} = P_1P_2 \cdot P_0P_6 = (0, 1, 2), & P_{11} = P_2P_5 \cdot P_0P_8 = (1, 1, 2), \\ P_{12} = P_0P_2 \cdot P_1P_7 = (2, 0, 1). & \end{array}$$

The lines are as follows:

$$\begin{array}{lll} p_0: x_3 = 0. & p_1: x_2 = 0. & p_2: x_1 + x_3 = 0. \\ p_3: x_2 + x_3 = 0. & p_4: x_1 + x_2 + x_3 = 0. & p_5: x_1 + x_2 - x_3 = 0. \\ p_6: -x_1 + x_2 + x_3 = 0. & p_7: x_1 + x_2 = 0. & p_8: x_1 - x_2 = 0. \\ p_9: x_2 - x_3 = 0. & p_{10}: x_1 - x_2 + x_3 = 0. & p_{11}: x_1 - x_2 = 0. \\ p_{12}: x_1 = 0. & & \end{array}$$

5. The points $P_0P_1P_2P_3P_4P_5P_6$ may be taken to be

$$(1, 0, 0) \quad (0, 1, 0) \quad (0, 0, 1) \quad (1, 1, 0) \quad (0, 1, 1) \quad (1, 1, 1) \quad (1, 0, 1).$$

§14.3

1. The points $(1, 0, 0)$, $(1, 1, 1)$, $(p, 1, 1)$ all lie on the line $x_2 = x_3$. We obtain $(0, q - 1, 1 - r)$ by subtracting $(1, 1, r)$ from $(1, q, 1)$.

3. $S = P_{11}$, $T = P_5$, $U = V = F = P_{10}$. (The point P_8 is not used.)

§14.4

1. The definition for a harmonic set involves A and B symmetrically, also C and F in the same way.

2. Draw any triangle RSP whose sides SP, PR, RS pass respectively through A, B, C. Let AR meet BS in Q. Then PQ meets AB in the desired harmonic conjugate.

3. Taking RAB as triangle of reference, let C and S be $(0, 1, \lambda)$ and $(1, 1, \lambda)$. Then Q is $(1, 1, 0)$, P is $(1, 0, \lambda)$, and F is $(1, 1, 0) - (1, 0, \lambda) = (0, 1, -\lambda)$.

4. On any line in $PG(2, 3)$, there are exactly four points. The harmonic conjugate of any one of these with respect to any two others is a fourth point on the line and therefore can only be *the* fourth point on the line!

5. The same harmonic set is determined projectively by the quadrangle, and affinely by dividing the segment AA' internally at A_1 and externally at A_2, in the same ratio.

§14.5

1. If x and x' are corresponding lines of two perspective pencils, their point of intersection $x \cdot x'$ continually lies on the axis o.

2. Any section of a harmonic set of lines is a harmonic set of points, and any harmonic set of points is projected by a harmonic set of lines.

5. Whenever a projectivity on a line g is the product of two perspectivities, the join of the two centers meets g in an invariant point.

7. If the given projectivity is an involution, say $(AA')(BB')$, it is expressible as the product of the two involutions $(AB)(A'B')$ and $(AB')(BA')$. If the given projectivity is not an involution, and A is any noninvariant point, the projectivity may be expressed as $AA'A'' \barwedge A'A''A'''$ (where possibly A''' coincides with A); it is then seen to be the product of the two involutions

$$(AA'')(A'A') \qquad \text{and} \qquad (AA''')(A'A'').$$

8. (i) $(c_{11} - c_{22})^2 + 4c_{12}c_{21} = 0$.
(ii) $c_{11} + c_{22} = 0$.

§14.6

1. Let AA' be the given pair of corresponding points, collinear with the center O. Let AX meet the axis in C. Then the collineation takes AC to $A'C$ and leaves invariant the line OX. Therefore X' is the point of intersection of $A'C$ and OX.

2. In the notation of Figure 14.3a, consider the perspective collineation with center O and axis DE that transforms P into P'. When the construction in ex. 1 is applied to Q it yields Q', and when it is applied to R it yields R'.

3. Let two points A and X, outside the line o of invariant points, be transformed into A' and X'. Since AA' and XX' are invariant lines, their common point O is an invariant point and therefore lies on o. Hence all joins of pairs of corresponding points meet o in the same point O.

4. In the notation of ex. 3, let O_1 be the harmonic conjugate of O with respect to A and A'. Then the harmonic homologies with centers A and O_1 will have the desired effect, since the former leaves A invariant and the latter takes A to A'.

5. Yes. For the unique projectivity $P_0P_1P_3 \barwedge P_1P_2P_4$ must transform the remaining point on P_0P_1 into the remaining point on P_1P_2. It is not necessary to give actual perspectivities, but in case they are desired, one possibility is

$$P_0P_1P_3P_9 \overset{P_{10}}{\underset{\wedge}{=}} P_0P_2P_8P_{12} \overset{P_9}{\underset{\wedge}{=}} P_1P_2P_4P_{10}.$$

$P_i \to P_{3i}$ is a projective collineation of period 3.

6. (i) A homology with center $(0, 0, 1)$ and axis $x_3 = 0$.
(ii) An elation with center $(c_1, c_2, 0)$ and axis $x_3 = 0$.

7. Consider a quadrilateral $APXA_1P_1X_1$, as in Figure 14.6*b*, with A conjugate to A_1 and P to P_1. The polars a and p pass through A_1 and P_1, respectively. By 14.64, the polar triangles APX and apx are perspective from the line A_1P_1. Therefore x passes through X_1, and X is conjugate to X_1.

8. The condition for the two points $(0, 1, \pm 1)$ to be conjugate is $c_{22} - c_{33} = 0$; for $(\pm 1, 0, 1)$, $c_{33} - c_{11} = 0$. These two conditions imply $c_{11} - c_{22} = 0$, which is the conjugacy condition for $(1, \pm 1, 0)$.

9. The given bilinear relation makes $x_1 = 0$ the polar of $(1, 0, 0)$, and $x_1 + x_2 + x_3 = 0$ the polar of $(1, 1, 1)$. Any self-conjugate point (x) must satisfy $x_1^2 + x_2^2 + x_3^2 = 0$. This is impossible in the real field but happens for all the four points $(1, \pm 1, \pm 1)$ in $PG(2, 3)$.

§14.7

2. When $B = D$, we have $x = p$, $y = PQ = d$, and $x \cdot y = P$.
When $A = D$, we have $y = q$, $x = d$, and $x \cdot y = Q$.

3. See Coxeter **2**, pp. 88–89.

7. The hint shows that the correlation $P_i \to p_i$ is projective. Being obviously of period 2, it is a polarity. The triangle $P_4P_{10}P_{12}$, whose sides are p_4, p_{10}, p_{12}, is self-polar. Finally, since the residues 0, 7, 8, 11 are the halves (mod 13) of 0, 1, 3, 9, the points P_0, P_7, P_8, P_{11} (and no others) lie on their polars.

Thus the four lines p_0, p_7, p_8, p_{11} are tangents, the six lines $p_1, p_2, p_3, p_5, p_6, p_9$ are secants, and the three lines p_4, p_{10}, p_{12} are non-secants. The three non-secants are the sides of the self-polar triangle $P_4P_{10}P_{12}$ which was used in describing the polarity. Since each non-secant is a common side of two self-polar triangles, there are three further self-polar triangles

$$P_4P_5P_9, \qquad P_{10}P_3P_6, \qquad P_{12}P_1P_2,$$

each having for its sides one non-secant and two secants [like the triangle EHH of Coxeter **2**, p. 82, Fig. 6.2C]. Each secant, containing only one pair of distinct conjugate points, is a side of only one self-polar triangle. Hence the only self-polar triangles are the four already mentioned.

Other geometries, such as $PG(2, 5)$, admit self-polar triangles formed by three secants or by one secant and two non-secants.

8. The sides of this hexagon are

$$x_1 = 0, \qquad x_1 = x_2 + x_3, \qquad x_2 = 0,$$
$$x_1 + x_3 = 2x_2, \qquad x_1 + x_2 = \tfrac{7}{5}x_3, \qquad \tfrac{1}{2}x_1 + x_2 = x_3.$$

Opposite sides meet in the three points

$$(0, 1, 2), \qquad (6, 1, 5), \qquad (2, 0, 1),$$

which all lie on the line $x_1 + 4x_2 = 2x_3$.

§14.8

1. Two distinct transversals from R would determine a plane containing both a and b.

2. Let a, b, c be three skew generators. Let an arbitrary plane through a meet c in R. This plane contains also the generator $Ra \cdot Rb$.

3. The four lines A_iB_i all intersect one another, and since they are not all coplanar they must be concurrent.

4. Calling the centers of perspective C_1, C_2, C_3, C_4, we see that $C_1C_2C_3C_4$ is perspective with $B_2B_3B_4B_1$ from A_1, with $B_1B_4B_3B_2$ from A_2, with $B_4B_1B_2B_3$ from A_3, with $B_3B_2B_1B_4$ from A_4, with $A_2A_3A_4A_1$ from B_1, and so on (with A's and B's consistently interchanged).

5. Each point of $PG(3, p)$ lies on $p^2 + p + 1$ lines, each containing p further points. Hence there are altogether $1 + p(p^2 + p + 1) = p^3 + p^2 + p + 1$ points, and, by duality, the same number of planes. Each of the $p^3 + p^2 + p + 1$ planes contains $p^2 + p + 1$ lines, but each line lies in $p + 1$ planes; therefore the total number of lines is

$$\frac{(p^3 + p^2 + p + 1)(p^2 + p + 1)}{p + 1} = (p^2 + 1)(p^2 + p + 1).$$

This expression was obtained by Von Staudt in 1856. (See the footnote on p. 237. See also P. H. Schoute, *Mehrdimensionale Geometrie*, vol. 1, p. 5, Leipzig, 1902.) When $p = 3$, the number is 130.

§14.9

1. Because Euclidean geometry does not admit self-perpendicular lines.

3. This is Clifford's first theorem in its original form, which can be derived from the form given in our text by inversion in a circle with center S.

§15.1

1. By 15.11 there is, on the ray AB, a point B' such that $CD \equiv AB'$. Thus we have $AB \equiv CD$ and $CD \equiv AB'$. By 15.12, $AB \equiv AB'$. But $AB \equiv AB$, and both B and B' are on the ray AB. Hence, by 15.11, $B' = B$.

2. Any triangle has an incircle, and the lengths of the tangents to it from A, B, C are $s - a$, $s - b$, $s - c$, as in §1.5, ex. 3. We have to abandon all the formulas involving trigonometrical functions, but ex. 1 remains valid. Even an acute-angled triangle may fail to have a circumcircle.

§15.2

1. See Coxeter **3**, p. 189.

2. If two lines have a common perpendicular m, they are symmetrical by reflection in m. Any point of intersection on one side of m would yield another on the opposite side, contradicting 12.2511.

§15.3

1. The plane through l perpendicular to the plane ABC meets the latter in a line m which may intersect l or be parallel or ultraparallel to l. In the first case, all the planes Al, Bl, Cl pass through the point of intersection. In the second case, they pass through the common end of l and m. In the third case, by 15.26, l and m have a common perpendicular EF, and the planes Al, Bl, Cl are all perpendicular to the plane through EF perpendicular to l.

2. By 15.23, p and r are parallel. Therefore the product of reflections in them is a parallel displacement. The first reflection leaves J invariant; the second transforms J into L.

§15.4

1. A tetragonal rotation about the front vertex on the left, a trigonal rotation about the center of the face d, the half-turn about the line joining the midpoints of two

opposite edges, and the half-turn about the line joining two opposite vertices: $(ab)(cd) = (acbd)^2$. Not counting the identity, we have $6 + 8 + 6 + 3 = 23$.

2. The cyclic group C_p has two "sets of poles," each consisting of just one pole. By adding an ineffective p_3-gonal pole with $p_3 = 1$, we are able to include this group as a "trivial" solution of 15.42.

§15.5

1. (a) C_2.
(b) $D_{2n}D_n$ (n even), $D_n \times \{\mathrm{I}\}$ (n odd).

2. C_6C_3.

§15.6

(a) $D_2 \times \{\mathrm{I}\}$. (b) $D_3 \times \{\mathrm{I}\}$. (c) $A_4 \times \{\mathrm{I}\}$.

§15.7

1. The vertices of $\{p, 2\}$ are p points evenly spaced along a great circle (say, the equator), and its edges are the p arcs that join neighboring vertices. The vertices of $\{2, p\}$ are two antipodal points (say, the north and south poles), and its edges are evenly spaced great semicircles (meridians).

2. The tetrahedron has six planes of symmetry, each joining an edge to the midpoint of the opposite edge. The cube and octahedron have nine, one parallel to each pair of opposite faces of the cube and one joining each pair of opposite edges. The dodecahedron and icosahedron have fifteen, one joining each pair of opposite edges.

3. $4\pi \Big/ \left(\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{2} - \pi\right)$. Two corresponding edges of the blown-up $\{p, q\}$ and $\{q, p\}$ perpendicularly bisect each other at the common right angle of four specimens of the fundamental region. Accordingly, it is natural that the order of the group should turn out to be $4E$.

§15.8

1. If the nth radius is k_n, we have $k_{n-1}k_{n+1} = k_n^2$.

3. Using the abbreviation

$$k^2 = \cos^2\frac{\pi}{p_1} - \sin^2\frac{\pi}{p_2} = \cos^2\frac{\pi}{p_2} - \sin^2\frac{\pi}{p_1}$$

(cf. 10.42), we find that the radius and distance are

$$\frac{1}{k}\sin\frac{\pi}{p_1} \qquad \text{and} \qquad \frac{1}{k}\cos\frac{\pi}{p_2}\,.$$

§16.1

If AB and AM, respectively, are perpendicular and parallel to r, as in Figure 16.3a, we have an acute angle at A and a right angle at B, and yet the rays do not meet.

§16.2

1. In the projective model, points and lines are represented by points and lines. Therefore isometries are represented by collineations. Since parallel lines are transformed into parallel lines, these collineations must transform ω into itself. Since a

reflection leaves invariant every point on the mirror, the corresponding collineation must be an elation or a homology (§14.6). Since it is of period 2, it can only be a harmonic homology. Since it preserves ω, its center is the pole of the mirror (i.e., the point of intersection of the tangents at the ends of the chord).

When the mirror is represented by one of the vertical circles, the reflection appears as the inversion in the sphere, through this circle, orthogonal to Klein's sphere.

2. A pencil of concentric circles may be described as the orthogonal trajectories of a pencil of lines. In the conformal model these lines appear as circles through a pair of inverse points (with respect to Ω). Therefore the circles belong to the orthogonal pencil of coaxal circles (including Ω and having the pair of inverse points for limiting points).

§16.3

1. In the projective model, the common perpendicular to two ultraparallel lines joins their poles with respect to ω, and the common parallel to two rays joins their ends.

2. For any point G on A/B, we have $\angle MAG > \angle MBA$. But $\angle DAM = \angle EBM$. Therefore $\angle DAG > \angle EBA = \angle BAD$.

3. By considering congruent right-angled triangles, we see that $AD = CF = BE$. Since the angle C of the triangle ABC is equal to $\angle CAD + \angle EBC$, the sum of all three angles of the triangle is

$$\angle BAD + \angle EBA,$$

that is, the sum of two (equal) acute angles.

4. This is a generalization of the theorem that the altitude lines of a triangle are concurrent (which is a corollary of Fagnano's problem). The simplest proof uses the projective model and refers to Chasles's theorem (14.64): *Any two polar triangles are perspective triangles.*

5. Either a translation or a glide reflection, according as the triangles are, or are not, on the same side of their common side.

6. To a circle of the indicated radius, draw tangents at the ends of three radii making angles 120° with one another. These tangents form a trebly asymptotic triangle.

7. Draw the Cevian through the given point and compare Figure 16.3*a*.

§16.5

3. Consider how successive translations along CA, AB, BC will affect the side CA of the triangle ABC. The first translation slides this segment CA along itself to a position AX. The second (along AB) takes this to BY, where $\angle ABY = A$. The third (along BC) takes this to CZ, where $\angle ZCB = \pi - \angle CBY = \pi - (A + B)$, so that $\angle ZCA = \pi - A - B - C$. (This result can evidently be extended from triangles to higher polygons.)

4. Consider how successive half-turns about the midpoints DA, AB, BC, CD will affect the side DA of the quadrangle $ABCD$. The first half-turn reverses this side, yielding AD. The second (about the midpoint of AB) takes this to BX, where $\angle ABX = A$. The third (about the midpoint of BC) takes this to CY, where $\angle BCY = A + B$. The fourth (about the midpoint of CD) takes this to DZ, where $\angle CDZ = A + B + C$, so that $\angle ZDA = 2\pi - A - B - C - D$.

5. At any vertex we find one specimen of each angle of the polygon, in natural

order. The cycle may be repeated any number of times (if the polygon has a sufficiently large area).

§16.6

1. Compare §16.3, ex. 3. The perpendicular bisectors of the sides of the triangle may be either intersecting or parallel or ultraparallel.

2. The external bisectors of two angles of the triangle may be either intersecting or parallel or ultraparallel.

3. The horocycle is symmetrical by reflection in any diameter. The diameter r reflects J into L.

4. Two. Their centers are the two ends of the perpendicular bisector of the segment joining the two points.

5. Remember that an equidistant curve has two branches.

7. Use §16.3, ex. 2.

§16.7

1. An equidistant curve.

2. The additive property, described in §6.6, ex. 5, shows that hyperbolic distance is *proportional* to inversive distance. The factor of proportionality is a matter of convention, like the value $\mu = 1$ that led to 16.53.

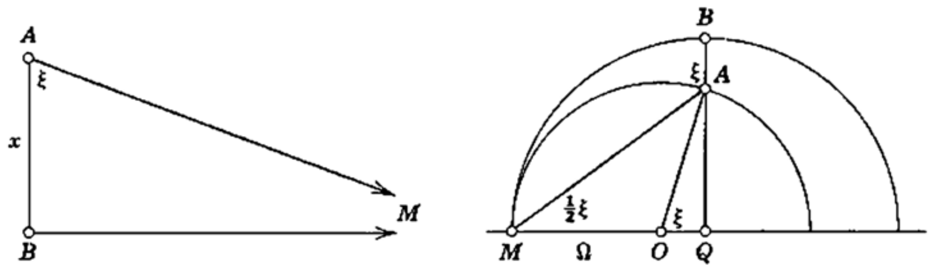


Figure 16.7a

3. The following proof of Lobachevsky's famous formula is credited to Paul Szász [see Coxeter, *Annali di Matematica, pura ed applicata*, (4), **71** (1966), p. 82]. In Figure 16.7*a*, the segment AB of length x is represented by the part AB of the line perpendicular to Ω at Q. The circle through B with center Q represents the line through B perpendicular to AB, and the tangent circle through A with center O (also on Ω) represents a parallel line having the same end M. The angle of parallelism

$$\xi = \Pi(x) = \angle BAM$$

appears as the angle at A in the "curvilinear triangle" BAM, and again as $\angle QOA$. Since x is the inversive distance between concentric circles with radii QB and QA (the latter not drawn), we have

$$e^x = \frac{QB}{QA} = \frac{QM}{QA} = \cot \tfrac{1}{2}\xi,$$

whence

$$\xi = 2 \arctan e^{-x}.$$

Any reader who dislikes using Euclidean trigonometry to obtain a hyperbolic result may turn to page 377 for a purely hyperbolic proof.

§16.8

By all the circles through one point (the point of contact).

§17.1

1. The vectors **a** and **c** must be either parallel to each other or both perpendicular to **b**.

2. $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [\mathbf{acd}]\mathbf{b} - [\mathbf{bcd}]\mathbf{a} = [\mathbf{abd}]\mathbf{c} - [\mathbf{abc}]\mathbf{d}$

3.
$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b}) &= (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b}) - (\mathbf{b} \cdot \mathbf{a})(\mathbf{a} \cdot \mathbf{b}) \\ &= |\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a}|^2|\mathbf{b}|^2 \cos^2 \theta \\ &= |\mathbf{a}|^2|\mathbf{b}|^2 \sin^2 \theta \\ &= |\mathbf{a} \times \mathbf{b}|^2.\end{aligned}$$

§17.2

The velocity is in the direction of the tangent. The acceleration is directed towards the center along the radius.

§17.3

1. $x = u - \sin u$, $y = -1 - \cos u$. This is, of course, a congruent cycloid.

2. $x = \cos u + u \sin u$, $y = \sin u - u \cos u$. This is a kind of spiral (but, of course, not an equiangular spiral).

3. Since $r = s \cos \varphi$.

§17.4

1. At the origin.

4. (a) $s = 4 \sin \psi$.

(b) $s = \frac{1}{2}l(\csc \psi \cot \psi + \log \tan \frac{1}{2}\psi)$.

5. Since $u = \log(\cosh u + \sinh u) = \log(\sec \psi + \tan \psi)$,

$$\int \sec \psi \, d\psi = \log(\sec \psi + \tan \psi) + C.$$

§17.5

$\rho = a \sinh u$. At the cusp it is zero: the curvature is infinite.

§17.6

1. Differentiating $\mathbf{r}^2 = 1$ twice, we obtain $\mathbf{r} \cdot \mathbf{t} = 0$ and $\mathbf{r} \cdot \mathbf{p} + \rho = 0$, whence

$$(\mathbf{r} + \rho\mathbf{p}) \cdot \mathbf{t} = (\mathbf{r} + \rho\mathbf{p}) \cdot \mathbf{p} = 0.$$

2. Because $\mathbf{t} \cdot \dfrac{d}{ds}(\mathbf{r} + \rho\mathbf{p}) = 0$.

§17.7

2. The helicoid $\dfrac{y}{x} = \tan \dfrac{z}{c}$.

3. $\dfrac{c}{a} = 1$.

§17.8

2. $\kappa = \tau = \dfrac{1}{3(1 + u^2)^2}$.

§17.9

1. $x = a\mu^u \cos u, \quad y = a\mu^u \sin u, \quad z = c\mu^u.$

3. The angle is arccot $(\sqrt{1 + c^2/a^2} \log \mu)$.

5. The cylinder based on an equiangular spiral.

§18.1

$$J[\mathbf{r}^1\mathbf{r}^2\mathbf{r}^3] = J\mathbf{r}^1 \cdot (\mathbf{r}^2 \times \mathbf{r}^3) = (\mathbf{r}_2 \times \mathbf{r}_3) \cdot (\mathbf{r}^2 \times \mathbf{r}^3)$$
$$= (\mathbf{r}_2 \cdot \mathbf{r}^2)(\mathbf{r}_3 \cdot \mathbf{r}^3) - (\mathbf{r}_3 \cdot \mathbf{r}^2)(\mathbf{r}_2 \cdot \mathbf{r}^3) = 1.$$

§18.2

2. $\mathbf{u} \cdot \mathbf{v} = g_{11}u^1v^1 + g_{22}u^2v^2 + g_{33}u^3v^3$
$$+ g_{12}(u^1v^2 + u^2v^1) + g_{23}(u^2v^3 + u^3v^2) + g_{31}(u^3v^1 + u^1v^3).$$

5. $\det g^{\alpha\beta} = G^{-1}$.

§18.3

2. $g_{\alpha\beta} = \delta_{\alpha\beta} + 1, \quad g^{\alpha\beta} = 4\delta^{\alpha\beta} - 1, \qquad$ where $\qquad \delta_{\alpha\beta} = \delta^{\alpha\beta} = \delta^{\alpha}_{\beta}$.

§18.5

3. $\Sigma x_\alpha{}^2 = 1, \quad \Sigma y_\alpha z_\alpha = 0.$

4. $g_{11} = g_{33} = 1, \quad g_{22} = (u^1)^2, \quad g_{\alpha\beta} = 0 (\alpha \neq \beta).$

5. $g_{11} = (u^3)^2, \quad g_{22} = (u^3 \sin u^1)^2, \quad g_{33} = 1, \quad g_{\alpha\beta} = 0\ (\alpha \neq \beta).$

6. $$g_{\alpha\beta} = \frac{1}{4}\left(\frac{x^2}{(A - u^\alpha)(A - u^\beta)} + \frac{y^2}{(B - u^\alpha)(B - u^\beta)} + \frac{z^2}{(C - u^\alpha)(C - u^\beta)}\right)$$

§18.6

1. $\Sigma\Sigma\epsilon^{\alpha\beta\gamma}y_\beta z_\gamma$.

§19.1

2. $\Sigma\mathbf{r}^j \times \mathbf{r}_j = \Sigma\Sigma g^{ij}\mathbf{r}_i \times \mathbf{r}_j = \sqrt{g}\ \Sigma\Sigma\, g^{ij}\epsilon_{ij}\mathbf{n} = \mathbf{0}.$

The triangle formed by $\mathbf{r}^1$ and $\mathbf{r}_1$ has the same area, apart from sign, as the triangle formed by $\mathbf{r}^2$ and $\mathbf{r}_2$.

3. $\mathbf{r}^1 = \mathbf{r}_1/(1 + z_1{}^2), \quad \mathbf{r}^2 = \mathbf{r}_2/(u^1)^2.$

4. $g_{11} = g^{11} = 1, \quad g_{22} = \sin^2 u^1,$
$g^{22} = \csc^2 u^1, \quad g_{ij} = g^{ij} = 0\ (i \neq j).$

§19.2

2. $\tan\phi = \sqrt{g}/g_{12}$.

4. $\cos\phi = \cos\theta\cos(\phi - \theta) - \sin\theta\sin(\phi - \theta)$
$$= \frac{a_1a_2}{g_1g_2} - \frac{a^2a^1}{g^2g^1}$$
$$= \frac{a_1}{g_1g_2}(g_{12}a^1 + g_{22}a^2) - \frac{a^2}{g^2g^1}(g^{11}a_1 + g^{12}a_2)$$
$$= \frac{g_{12}}{g_1g_2}(a_1a^1 + a^2a_2) + \left(\frac{g_2}{g_1} - \frac{g^1}{g^2}\right)a_1a^2$$
$$= \frac{g_{12}}{g_1g_2}.$$

5. $g_1 a^1 = g_2 a^2$.

6. The net consists of the parametric curves, which are orthogonal if $g^{12} = 0$. The identity $g^{11}g_{11} - g^{22}g_{22} = 0$ is in agreement with the fact that the internal and external bisectors of an angle are perpendicular.

7. $S = \int_0^{\pi}\int_0^{2\pi} \sin u^1 \, du^1 \, du^2$.

§19.3

1. $\mathbf{r}^1 \times \mathbf{r}^2 = \mathbf{n}/\sqrt{g}$.

2. $b_{11} = -1, \quad b_{22} = -\sin^2 u^1, \quad b_{ij} = 0 \ (i \neq j)$.

§19.4

1. $H = 0$.

2. $H = \{z_1(1 + z_1^2) + u^1 z_{11}\}/2u^1(1 + z_1^2)^{\frac{3}{2}}$.

3. $u^1 = \frac{1}{2}\pi, \quad u^1 = \frac{3}{2}\pi$.

5. At an umbilic.

§19.5

3. $u^1 \pm u^2 = k$.

4. $u^1 = \pm c \sinh (u^2 - k)$.

§19.6

1. At an umbilic, 19.52 is an identity.

2. The expression is a perfect square.

3. At an umbilic, $K = \kappa^2 > 0$.

4. The conditions $b_{11}:b_{12}:b_{22} = g_{11}:g_{12}:g_{22}$ become

$-\sin^2 u^2 : \sin u^1 \cos u^1 \sin u^2 \cos u^2 : -\sin^2 u^1$

$= 2\sin^2 u^1 + \cos^2 u^1 \sin^2 u^2 : \sin u^1 \cos u^1 \sin u^2 \cos u^2 : 2\sin^2 u^2 + \sin^2 u^1 \cos^2 u^2$.

5. No. When there is a curve of umbilics, this curve is itself a line of curvature, and the only lines of curvature that cross it do so at right angles.

§19.7

1. $b_{33} = 0$.

3. The lines of curvature are the intersections of the ellipsoid with the other quadrics of the system.

§19.8

1. $\theta = \frac{1}{4}\pi, \quad \frac{3}{4}\pi$.

§20.1

1. $\sqrt{g}\, K = \dfrac{\partial}{\partial u^1}\left(\dfrac{\sqrt{g}}{g_{22}} \Gamma^1_{22}\right) - \dfrac{\partial}{\partial u^2}\left(\dfrac{\sqrt{g}}{g_{22}} \Gamma^1_{12}\right)$.

$\Gamma_{ij,i} = \frac{1}{2}(g_{ii})_j, \quad \Gamma^k_{ij} = \Gamma_{ij,k}/g_{kk}$.

2. $\Gamma_{ii,i} = \frac{1}{2}(g_{ii})_i, \ \Gamma_{ii,j} = -\frac{1}{2}(g_{ii})_j, \quad \Gamma_{ij,i} = \frac{1}{2}(g_{ii})_j, \quad \Gamma^k_{ij} = \Gamma_{ij,k}/g_{kk}$.

3. $\Gamma_{12,2} = u^1, \quad \Gamma_{22,1} = -u^1, \quad \Gamma^2_{12} = 1/u^1, \quad \Gamma^1_{22} = -u^1$; all others are 0.

5. By the equations just before 19.33,

$$\sqrt{g}\,\mathbf{r}^i = \Sigma\epsilon^{hi}\mathbf{n} \times \mathbf{r}_h.$$

Hence
$$\begin{aligned}\sqrt{g}\Sigma\Gamma^i_{ij} &= \sqrt{g}\Sigma\mathbf{r}^i \cdot \mathbf{r}_{ij} = \Sigma\Sigma\epsilon^{hi}[\mathbf{n}\mathbf{r}_h\mathbf{r}_{ij}] \\ &= \mathbf{n} \cdot \Sigma\Sigma\epsilon^{hi}\mathbf{r}_h \times \mathbf{r}_{ij} = \mathbf{n} \cdot (\mathbf{r}_1 \times \mathbf{r}_2)_j \\ &= \mathbf{n} \cdot (\sqrt{g}\,\mathbf{n})_j = (\sqrt{g})_j.\end{aligned}$$

6. $K = 1$.

§20.2

3. No; the tangents do not make a constant angle with the z-axis.

§20.3

2. $(\sqrt{g})_{11} = -\sqrt{g}$.

§20.4

2. Another circle of radius b and one of radius $a - b$.

§20.5

(i) $2\pi \sin r$, (ii) $2\pi \sinh r$.

§20.7

$\cosh^2 u^1 + (u^2 + c)^2 = k^2$.

§21.1

Yes; it forms a map of three hexagons on the torus.

§21.2

2. Let $P_1P_2P_3P_4P_5P_6$ be a regular hexagon concentric with and interior to the disk. Join P_1P_4, P_2P_5, P_3P_6 through the boundary of the disk rather than through the center.

3. They form the Thomsen graph.

4. Yes, if $q > 0$. Both are nonorientable with $\chi = 2 - 2p - q$.

§21.3

1. $\{2, 1\}$ may be drawn as a great semicircle joining 2 antipodal vertices. $\{1, 2\}$ may be drawn as a great circle with one point on it specified as a vertex.

2. $\{3, 5\}/2$ has 6 vertices, each joined to every other. The vertices and edges of $\{5, 3\}/2$ form the Petersen graph [Ball **1**, p. 225].

5. $\{4, 4\}_{1,1}$ has 2 quadrangular faces, 4 edges, and 2 vertices; each vertex belongs to all 4 edges.

$\{4, 4\}_{2,0}$ has 4 quadrangular faces, 8 edges, and 4 vertices; each vertex belongs to all 4 faces.

$\{3, 6\}_{1,1}$ has 6 triangular faces, 8 edges, and 3 vertices; each vertex belongs to all 6 faces.

$\{6, 3\}_{1,1}$ has 3 hexagonal faces, 9 edges, and 6 vertices; each vertex belongs to all 3 faces.

$\{6, 3\}_{2,0}$ has 4 hexagonal faces, 12 edges, and 8 vertices.

All 5 maps are of genus 1.

7. $V = 3, 4, 4, 5, 6, 7$.

9. The positive integers p and q are not quite arbitrary. If one of them is 1, the other can only be 2. For instance, $q = 1$ imples $E = pr$, $F = 2r$, whence

$$E + F = (p + 2)r = (2p + 2q - pq)r = \chi \leqslant 2,\ E = F = 1,\ p = 2.$$

§21.4

1. The cube in only one way; the dodecahedron two ways.

§21.6

$$\begin{aligned} \chi &= 2,\ 1,\ 0,\ -1,\ -2,\ -3,\ -4,\ -5,\ -6,\ -7,\ -8,\ -9;\\ [N] &= 4,\ 6,\ 7,\ 7,\ 8,\ 9,\ 9,\ 10,\ 10,\ 10,\ 11,\ 11. \end{aligned}$$

§22.1

1. $rN_1 = E'N_0$, where, by ex. 1 at the end of §10.3,

$$\frac{1}{E'} = \frac{1}{q} + \frac{1}{r} - \frac{1}{2}.$$

Similarly, $pN_2 = EN_3$.

2. This is derived from the analogous cube in the space $x_1 = 0$ by translating it through distance 1 along the fourth dimension.

3. The cube $(\pm 1, \pm 1, \pm 1, -1)$ is translated through distance 2 along the fourth dimension.

4. $(1, 1, 1, 1)$.

§22.2

1. Since $\cos\frac{\pi}{q} < \sin\frac{\pi}{p}$, $\frac{\pi}{p} + \frac{\pi}{q} > \frac{\pi}{2}$. Similarly, $\cos\frac{\pi}{q} < \sin\frac{\pi}{r}$.

2. $\{3, 3, 4\}$.

§22.3

1. $(\pm 1, 0, 0, 0)$, $(0, \pm 1, 0, 0)$, $(0, 0, \pm 1, 0)$, $(0, 0, 0, \pm 1)$.

2. $(\pm 1, \pm 1, 0, 0)$, permuted.

3. $(\tau, 1, \tau^{-1}, 0) = \tau^{-2}(\tau, 0, \tau, 0) + \tau^{-1}(\tau, \tau, 0, 0)$.

4. The extra points correspond to the centers of the 24 icosahedra.

§22.4

1. No.

2. (a) No. (b) Yes.

3. Yes. The twelve can have their centers at the vertices of a regular icosahedron.

4. $1^2 + 2^2 + \cdots + n^2 = n(n + 1)(2n + 1)/6$.

$$\binom{2}{2} + \binom{3}{2} + \cdots + \binom{n+1}{2} = \binom{n+2}{3}.$$

§22.5

No.

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